

CORRECTIONS TO “HOMOTOPY THEORY OF NONSYMMETRIC OPERADS, I, II”

FERNANDO MURO

ABSTRACT. We correct a mistake in the construction of push-outs along free morphisms of algebras over a nonsymmetric operad in [8], and we fix the affected results in [8, 9].

INTRODUCTION

In [8, §8] we give a wrong construction of push-outs along free maps in the category of algebras over an operad (nonsymmetric). A counterexample provided by Donald Yau is presented below. Our operads live in a symmetric monoidal category \mathcal{V} , but algebras live in a possibly nonsymmetric monoidal category \mathcal{C} with a central action of \mathcal{V} via a strong monoidal functor $z: \mathcal{V} \rightarrow \mathcal{C}$ equipped with some extra structure [8, §7]. This situation generalizes the usual *symmetric case*¹, where $\mathcal{C} = \mathcal{V}$ and z is the identity. However, contrary to what we intended and claimed in the introduction of [8], [8, §8] does not generalize Harper’s construction [4, Proposition 7.32], which is the correct one. We here fix this mistake and its consequences in [8, 9]. The main results of these papers, presented in their introductions, remain true as stated modulo a modification in the nonsymmetric monoid axiom [8, Definition 9.1] and a strengthening in the hypotheses of [9, Theorem 1.13 and Corollary 1.14]. These changes do not affect applications. Moreover, the results which are purely on operads, not on algebras, remain completely unaffected.

The corrections are presented in Section 1. The required proofs, in Section 3, are based in the categorical constructions of Section 2. As a byproduct, given a cofibrant operad \mathcal{O} in \mathcal{V} , we prove homotopy invariance for enveloping operads of \mathcal{O} -algebras in \mathcal{V} with underlying cofibrant object. This result, which is of independent interest, has been obtained in [3, §17.4] for symmetric operads in the category \mathcal{V} of chain complexes over a commutative ring.

1. CORRECTIONS

Consider the following push-out diagram in the category of algebras in \mathcal{C} over a certain operad \mathcal{O} in \mathcal{V} , where the top arrow is a free \mathcal{O} -algebra map,

$$(1-1) \quad \begin{array}{ccc} \mathcal{F}_{\mathcal{O}}(Y) & \xrightarrow{\mathcal{F}_{\mathcal{O}}(f)} & \mathcal{F}_{\mathcal{O}}(Z) \\ g \downarrow & \text{push} & \downarrow g' \\ A & \xrightarrow{f'} & B \end{array}$$

¹This does not refer to symmetric operads, which are not considered here. The symmetric case would be, more generally, when \mathcal{C} and z are symmetric monoidal, but this apparently more general case can be reduced to the former.

The adjoints of g and g' are maps $\bar{g}: Y \rightarrow A$ and $\bar{g}': Z \rightarrow A$ in \mathcal{C} , respectively.

The constructions in [8, Lemmas 8.1 and 8.2], contrary to what we claimed in [8, Theorem 8.3], do not yield an \mathcal{O} -algebra. The origin of the mistake is in the map [8, (15)], which must be replaced with a quotient in the category $\text{Mor}(\mathcal{C})$ of morphisms in \mathcal{C} [8, §4]. This quotient is related to the *enveloping functor-operad* \mathcal{O}_A of the \mathcal{O} -algebra A , see Section 2. A (nonsymmetric) *functor-operad* $F = \{F(n)\}_{n \geq 0}$ or *multitensor* [1] in \mathcal{C} is a sequence of functors $F(n): \mathcal{C}^n \rightarrow \mathcal{C}$ equipped with composition and unit natural transformations

$$\begin{aligned} \circ_i: F(p)(\cdot \cdot^{i-1} \cdot \cdot, F(q), \cdot \cdot^{p-i} \cdot \cdot) &\longrightarrow F(p+q-1), \quad 1 \leq i \leq p, q \geq 0, \\ u: \text{id}_{\mathcal{C}} &\longrightarrow F(1), \end{aligned}$$

satisfying relations similar to operads. In the following corrected statement of [8, Lemma 8.1] we simply use the functor-operad structure and the fact that $\mathcal{O}_A(0) = A$ (a functor $\mathcal{C}^0 \rightarrow \mathcal{C}$ is a plain object in \mathcal{C} since \mathcal{C}^0 is the final category).

Recall from [8, §4] that a map $f: Y \rightarrow Z$ in \mathcal{C} is the same as a functor $f: \mathbf{2} \rightarrow \mathcal{C}$ from the poset $\mathbf{2} = \{0 < 1\}$. Given a functor $F: \mathcal{C}^n \rightarrow \mathcal{C}$ and maps $f_i: Y_i \rightarrow Z_i$, $1 \leq i \leq n$, in \mathcal{C} , $F(f_1, \dots, f_n)$ usually denotes the induced map $F(Y_1, \dots, Y_n) \rightarrow F(Z_1, \dots, Z_n)$. However, we also denote by $F(f_1, \dots, f_n)$ the composite functor

$$\mathbf{2}^n \xrightarrow{f_1 \times \dots \times f_n} \mathcal{C}^n \xrightarrow{F} \mathcal{C}.$$

Moreover, unless otherwise indicated, in this paper $F(f_1, \dots, f_n)$ denotes its latching map at $(1, \dots, 1) \in \mathbf{2}^n$.

Lemma 8.1. *There is a sequence in \mathcal{C}*

$$A = B_0 \xrightarrow{\varphi_1} B_1 \rightarrow \dots \rightarrow B_{t-1} \xrightarrow{\varphi_t} B_t \rightarrow \dots,$$

such that the morphism φ_t , $t \geq 1$, is given by the push-out square

$$\begin{array}{ccc} \bullet & \xrightarrow{\mathcal{O}_A(t)(f, \dots, f)} & \bullet \\ \psi_t \downarrow & \text{push} & \downarrow \bar{\psi}_t \\ B_{t-1} & \xrightarrow{\varphi_t} & B_t \end{array}$$

where the attaching map ψ_t is defined by the following maps, $1 \leq i \leq t$,

$$\begin{aligned} &\mathcal{O}_A(t)(Z, \cdot \cdot^{i-1} \cdot \cdot, Z, Y, Z, \cdot \cdot^{t-i} \cdot \cdot, Z) \\ &\quad \downarrow \text{defined by } \bar{g}: Y \rightarrow A \\ &\mathcal{O}_A(t)(Z, \cdot \cdot^{i-1} \cdot \cdot, Z, A, Z, \cdot \cdot^{t-i} \cdot \cdot, Z) \\ &\quad \downarrow \circ_i \\ &\mathcal{O}_A(t-1)(Z, \cdot \cdot^{t-1} \cdot \cdot, Z) \\ &\quad \downarrow \bar{\psi}_{t-1} \text{ if } t > 1 \quad \downarrow \text{the identity if } t = 1 \\ &B_{t-1} \end{aligned}$$

There is a canonical map $[8, (15)] \rightarrow \mathcal{O}_A(t)(f, \dots, f)$ in $\text{Mor}(\mathcal{C})$, actually a projection onto a coequalizer, see Section 2. The map [8, (15)] is a coproduct indexed by the integers $n \geq 1$ and the subsets $S \subset \{1, \dots, n\}$ of cardinality t .

Hence, the vertical composites in the following diagram

$$\begin{array}{ccc}
 & \xrightarrow{[8, (15)]} & \\
 (\psi_t^{n,S})_{n,S} \swarrow & \begin{array}{c} \text{canonical map} \\ \mathcal{O}_A(t)(f, \dots, f) \end{array} \downarrow & \searrow (\bar{\psi}_t^{n,S})_{n,S} \\
 & \xrightarrow{\text{push}} & \\
 \downarrow \psi_t & & \downarrow \bar{\psi}_t \\
 B_{t-1} & \xrightarrow{\quad} & B_t
 \end{array}$$

are defined by their restrictions. With this notation, the rest of results in [8, §8] are correct as stated. The same proof as in the symmetric case [4, Proposition 7.32] works here, modulo slight changes in notation (Harper works in the more general context of left \mathcal{O} -modules instead of \mathcal{O} -algebras). In that case, the enveloping functor-operad is simply given by $\mathcal{O}_A(n)(X_1, \dots, X_n) = \mathcal{O}_A(n) \otimes \bigotimes_{i=1}^n X_i$, where \mathcal{O}_A on the right denotes the enveloping operad of A . The order of factors is also important in our case, but fortunately Harper’s notation, using symmetric groups, takes care of this. We think this is because the natural setting for this kind of result is precisely our nonsymmetric context. In the nonsymmetric case, $\Sigma_{p+q}/\Sigma_p \times \Sigma_q$ should be interpreted as the set of (p, q) -shuffles (each coset is represented by a unique (p, q) -shuffle).

Example 1.1. The following counterexample to the constructions in [8, §8] is due to Donald Yau, to whom we are very grateful. We place ourselves in the symmetric case, taking \mathcal{V} to be the category of complexes over a commutative ring and $\mathcal{O} = \mathbf{uAss}^{\mathcal{V}}$ the unital associative operad. Let $A = 0$ be the final $\mathbf{uAss}^{\mathcal{V}}$ -algebra and $f: 0 = Y \rightarrow Z$ a map from the zero complex. According to [8, §8], the push-out of A along the free $\mathbf{uAss}^{\mathcal{V}}$ -algebra map spanned by f is $\bigoplus_{t \geq 1} Z^{\otimes t}$. The real push-out is A again, since this $\mathbf{uAss}^{\mathcal{V}}$ -algebra has the unusual property that any morphism $A \rightarrow B$ must be an isomorphism.

Actually, the statements in [8, §8] do not yield Schwede–Shipley’s construction [10] of push-outs along free maps when $\mathcal{O} = \mathbf{uAss}^{\mathcal{V}}$, but the corrected version does, see Proposition 2.2 below.

In [8, 9], we derived some results from the homotopical properties of [8, (15)], which are nice and follow easily from the push-out product axiom. We should use $\mathcal{O}_A(t)(f, \dots, f)$ instead, whose homotopical properties are unfortunately worse and more difficult to establish. The first affected result is [8, Proposition 9.2], where we must modify the statement of (2). The full correct statement is the following.

Proposition 9.2. *Consider the push-out diagram (1-1) in $\text{Alg}_{\mathcal{C}}(\mathcal{O})$.*

- (1) *If f is a trivial cofibration in \mathcal{C} then the underlying morphism $f': A \rightarrow B$ in \mathcal{C} is a relative K' -cell complex, where K' is the class in Definition 9.1.*
- (2) *Suppose that f is a cofibration in \mathcal{C} and that one of the following statements holds:*
 - (a) *A is a cofibrant \mathcal{O} -algebra and $\mathcal{O}(n)$ is cofibrant in \mathcal{V} for $n \geq 0$.*
 - (b) *\mathcal{V} has cofibrant tensor unit, \mathcal{O} is a cofibrant operad or there is a cofibration $\mathbf{Ass}^{\mathcal{V}} \rightarrow \mathcal{O}$ or $\mathbf{uAss}^{\mathcal{V}} \rightarrow \mathcal{O}$ in $\text{Op}(\mathcal{V})$, and the \mathcal{O} -algebra A in \mathcal{C} has underlying cofibrant object.*

Then the morphism $f': A \rightarrow B$ is a cofibration in \mathcal{C} .

Although the statement of Proposition 9.2 (1) remains as in [8], we must modify [8, Definition 9.1] in the following way, in order to obtain it as a trivial consequence of [8, §8] (corrected version).

Definition 9.1. The *monoid axiom* in the \mathcal{V} -algebra \mathcal{C} says that relative K' -cell complexes are weak equivalences, where K' is the following class of morphisms,

$$K' = \{f \otimes X, X \otimes f, \mathcal{O}_A(t)(f, \dots, f); X \text{ is an object in } \mathcal{C}, f \text{ is a trivial cofibration in } \mathcal{C}, \mathcal{O} \text{ is an operad in } \mathcal{V}, A \text{ is an } \mathcal{O}\text{-algebra in } \mathcal{C}, t \geq 1\}.$$

This nonsymmetric monoid axiom is apparently stronger than the one in [8], but still equivalent to Schwede–Shipley’s in the symmetric case, since $X \otimes f \cong f \otimes X$ and $\mathcal{O}_A(t)(f, \dots, f) = \mathcal{O}_A(t) \otimes f^{\odot t}$, $t \geq 1$, where $\mathcal{O}_A(t)$ on the right is an object and $f^{\odot t}$ is a trivial cofibration by the push-out product axiom. Moreover, it holds in our main nonsymmetric example, the category $\text{Graph}_S(\mathcal{V})$ of \mathcal{V} -graphs with object set S , i.e. [8, Proposition 10.3] remains valid. A new proof is required though, but the argument is very similar. It is easy to check (using the symmetry of \mathcal{V}) that any map in K' is componentwise a coproduct of maps, each of which is the tensor product of a single object in \mathcal{V} with a push-out product of components of f , which are trivial cofibrations in \mathcal{V} . Such a push-out product is again a trivial cofibration by the push-out product axiom. Hence, any K' -cell complex is componentwise a K -cell complex in the sense of [8, Definition 6.1], and therefore a weak equivalence by the monoid axiom for \mathcal{V} .

Proposition 9.2 (2) (a) is the arity 0 part of Proposition 3.5, and (b) follows from Propositions 3.9, 3.15, and 3.10. Conditions (a) and (b) are however not necessary for f' to be a cofibration. For instance, in the symmetric case, suppose that all objects in \mathcal{V} are cofibrant. Then, for any operad \mathcal{O} and any \mathcal{O} -algebra A , $\mathcal{O}_A(t) \otimes f^{\odot t}$ is a cofibration for all $t \geq 1$ by the push-out product axiom. Hence f' is a transfinite composition of cofibrations in \mathcal{V} by [8, §8] (corrected version). The same holds for $\mathcal{C} = \text{Graph}_S(\mathcal{V})$. Nevertheless, we can give the following counterexample to the original statement of [8, Proposition 9.2 (2)].

Example 1.2. Assume we are in the symmetric case with \mathcal{V} the category of chain complexes over a commutative ring \mathbb{k} . Let \mathcal{O} be the operad whose algebras are non-unital DG-algebras A with $A^3 = 0$. This operad has a presentation with a degree 0 generator $\mu \in \mathcal{O}(2)$ and two arity 3 relations $\mu \circ_1 \mu = 0 = \mu \circ_2 \mu$. Moreover, $\mathcal{O}(1) = \mathbb{k} \cdot u$, $\mathcal{O}(2) = \mathbb{k} \cdot \mu$, and $\mathcal{O}(n) = 0$ otherwise, so \mathcal{O} is aritywise cofibrant. The enveloping operad of an \mathcal{O} -algebra A is $\mathcal{O}_A(0) = A$, $\mathcal{O}_A(1) = \mathbb{k} \oplus A/A^2 \oplus A/A^2$, $\mathcal{O}_A(2) = \mathbb{k}$, and 0 otherwise, see Proposition 2.5 below. In general, $A \amalg \mathcal{F}_{\mathcal{O}}(\mathbb{k}) = \bigoplus_{n \geq 0} \mathcal{O}_A(n)$ is the direct sum of the components of the enveloping operad, see [4, Proposition 7.28], and the inclusion of the first factor $f': A \hookrightarrow A \amalg \mathcal{F}_{\mathcal{O}}(\mathbb{k})$ is the inclusion of the first direct summand $\mathcal{O}_A(0) = A$. This inclusion is an \mathcal{O} -algebra cofibration since it is f' in (1-1) for $f: 0 \rightarrow \mathbb{k}$.

Let $\mathbb{k} = \mathbb{Z}$ and A be the \mathcal{O} -algebra with two degree 0 generators x, y satisfying the relations $x^2 = 2y, xy = yx = y^2 = 0$. This algebra is $A = \mathbb{Z} \cdot x \oplus \mathbb{Z} \cdot y$ concentrated in degree 0, so it is cofibrant as a complex. The cokernel of f' in \mathcal{V} contains $\mathcal{O}_A(1)$, which is not free since $A/A^2 = \mathbb{Z} \oplus \mathbb{Z}/2$, so f' is not a cofibration in \mathcal{V} .

The modifications made in [8, Proposition 9.2] have no impact on [8, Lemma 9.4 and Corollary 9.5], however [8, Lemma 9.6 and Theorem 1.3] require a new proof.

They follow from the arity 0 part of Proposition 3.6. Actually, [9, Theorem 6.7], which is [8, Theorem 1.3] with one less hypothesis, follows in this way. Similarly [9, Theorem D.4], which generalizes [9, Theorem 6.7], see Remark 3.16. The rest of results in [8] are unaffected by the corrections. We now move to [9].

In [9, Remark 6.4] we give a short description of the results in [8, §8]. Since we have corrected [8, Lemma 8.1], we must modify [9, Remark 6.4] accordingly. More precisely [9, (6-2)] should be replaced with $\mathcal{O}_A(t)(f, \dots, f)$. This modification forces us to give a new proof of [9, Proposition 7.3]. It is the arity 0 part of Proposition 3.7. We must replace the monoid axiom in [9, Definition 2.3 (3)] with the new version of [8, Definition 9.1] above. In addition, the following new hypothesis should be added to the indicated results.

New Hypothesis ([9, Theorems 1.13, 8.1, and D.13, Corollaries 1.14 and 8.2, and Propositions 8.3 and D.14]). *The operad \mathcal{O} is cofibrant in $\text{Op}(\mathcal{V})$ or admits a cofibration from the associative operad $\mathbf{Ass}^{\mathcal{V}} \rightarrow \mathcal{O}$ or from the unital associative operad $\mathbf{uAss}^{\mathcal{V}} \rightarrow \mathcal{O}$.*

The most general of these results is [9, Theorem D.13], which follows from Theorem 3.11, Propositions 3.9 and 3.15, and Remark 3.16 below. The remark is not required for the results which are not in [9, Appendix D], provided we also assume that the tensor unit of \mathcal{V} and $\mathcal{O}(n)$, $n \geq 0$, are cofibrant. Note that [9, Lemmas 6.6 and D.1] are not useful any more, since the map [9, (6-2)] plays no role after the corrections. Finally, [9, Corollary D.2], which generalizes [8, Proposition 9.2 (2)], should be similarly corrected.

Corollary D.2. *Suppose that \mathcal{C} satisfies the strong unit axiom and that one of the following statements holds:*

- (1) *A is a cofibrant \mathcal{O} -algebra and $z(\mathcal{O}(n))$ is pseudo-cofibrant in \mathcal{C} for $n \geq 0$.*
- (2) *\mathcal{O} is a cofibrant operad or there is a cofibration $\mathbf{Ass}^{\mathcal{V}} \rightarrow \mathcal{O}$ or $\mathbf{uAss}^{\mathcal{V}} \rightarrow \mathcal{O}$ in $\text{Op}(\mathcal{V})$, and the \mathcal{O} -algebra A in \mathcal{C} has underlying pseudo-cofibrant object.*

Then any cofibration $\phi: A \rightarrow B$ in $\text{Alg}_{\mathcal{C}}(\mathcal{O})$ is also a cofibration in \mathcal{C} .

The proof follows the same steps as the proof of Proposition 9.2 (2) above, considering also Remark 3.16, and [9, Corollary D.3] is an immediate consequence. There are no more affected results in [9].

Example 1.3. We here give a counterexample to the original [9, Theorem 1.13] for \mathcal{V} the category of complexes over a field \mathbb{k} . It can also be used to disprove the other results where the new hypothesis is required. Let us place ourselves in the context of Example 1.2. Let $B = \mathbb{k} \cdot x$ be the \mathcal{O} -algebra concentrated in degree 0 with $x^2 = 0$, and A the \mathcal{O} -algebra concentrated in degrees 1 and 0

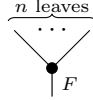
$$\cdots \rightarrow 0 \rightarrow \mathbb{k} \cdot z \rightarrow \mathbb{k} \cdot y \oplus \mathbb{k} \cdot y^2 \rightarrow 0 \rightarrow \cdots$$

with $yz = zy = z^2 = 0$ and $d(z) = y^2$. There is an obvious weak equivalence of \mathcal{O} -algebras $\varphi: A \xrightarrow{\sim} B$ defined by $\varphi(y) = x$. If $\psi: A \rightarrow A \amalg \mathcal{F}_{\mathcal{O}}(\mathbb{k}) = C = \bigoplus_{n \geq 0} \mathcal{O}_A(n)$ is the inclusion of the first factor of the coproduct then so is $\psi': B \rightarrow B \amalg \mathcal{F}_{\mathcal{O}}(\mathbb{k}) = B \cup_A C = \bigoplus_{n \geq 0} \mathcal{O}_B(n)$. However, $\varphi': C \rightarrow B \cup_A C$ cannot be a weak equivalence since \mathcal{O}_B is concentrated in degree 0 but $\mathcal{O}_A(1) = \mathbb{k} \oplus A/A^2 \oplus A/A^2$ and $H_1(A/A^2) = \mathbb{k} \cdot z$.

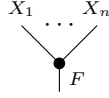
2. ENVELOPING FUNCTOR-OPERADS

Given arbitrary categories \mathcal{M} and \mathcal{N} , we will work with the big categories $\mathcal{M}^{\mathcal{N}^n}$ of functors $\mathcal{N}^n \rightarrow \mathcal{M}$ and natural transformations between them, $n \geq 0$. All invoked categorical notions in $\mathcal{M}^{\mathcal{N}^n}$ will tacitly have the pointwise meaning. In this way, we will not incur in any contradiction derived from the fact that morphism classes in $\mathcal{M}^{\mathcal{N}^n}$ may not be sets. We will also work with big categories of sequences of functors $\mathcal{M}^{\mathcal{N}^{(\mathbb{N})}} = \prod_{n \geq 0} \mathcal{M}^{\mathcal{N}^n}$.

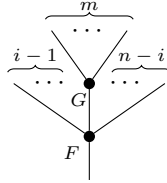
A functor $F: \mathcal{C}^n \rightarrow \mathcal{C}$, $n \geq 0$, will be denoted by a corolla with n leaves with inner vertex labeled with F ,



The evaluation $F(X_1, \dots, X_n)$ of this functor at n objects X_1, \dots, X_n in \mathcal{C} will be designated by labeling the leaves with these objects



If $G: \mathcal{C}^m \rightarrow \mathcal{C}$, $m \geq 0$, is another functor and $1 \leq i \leq n$, composition at the i^{th} slot $F \circ_i G = F(\cdot \overset{i-1}{\dots}, G, \cdot \overset{n-i}{\dots})$: $\mathcal{C}^{n+m-1} \rightarrow \mathcal{C}$ will be denoted by tree grafting

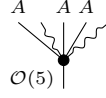


In this way, a *tree* (planted, planar and with leaves) in the sense of [8, §3], where each vertex v is labeled with a functor $\mathcal{C}^{\tilde{v}} \rightarrow \mathcal{C}$ (\tilde{v} is the *arity* of v), denotes a functor $\mathcal{C}^n \rightarrow \mathcal{C}$, where n is the number of leaves, and evaluation of this functor is designated by labeling the leaves with objects in \mathcal{C} .

We say that a set of inner vertices in a tree is labeled with an object $F = \{F(n)\}_{n \geq 0}$ in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ if each vertex v in the set is labeled with $F(\tilde{v})$. A sequence $\{V(n)\}_{n \geq 0}$ of objects in \mathcal{V} is regarded as the sequence of functors $(X_1, \dots, X_n) \mapsto z(V(n)) \otimes \bigotimes_{i=1}^n X_i$. In this way, we can regard any operad in \mathcal{V} as a functor-operad.

Some labeled trees below have leaves decorated with shapes. This does not change the functorial meaning of labeled trees, it is only used to define appropriate indexing sets and to indicate the labels.

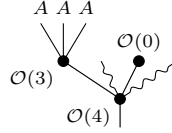
Let \mathcal{O} be an operad in \mathcal{V} and A an \mathcal{O} -algebra in \mathcal{C} . Let \mathcal{O}_A^0 be the sequence of functors such that $\mathcal{O}_A^0(n)$ is the coproduct of all corollas equipped with n distinguished snaky leaves, inner vertex labeled with \mathcal{O} , and straight leaves with A , e.g. ($n = 2$)



The map [8, (15)] is $\mathcal{O}_A^0(t)(f, \dots, f)$. We must replace it with $\mathcal{O}_A(t)(f, \dots, f)$, where \mathcal{O}_A is the reflexive coequalizer in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ of a diagram

$$(2-1) \quad \mathcal{O}_A^1 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{O}_A^0 .$$

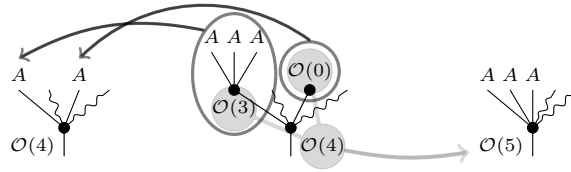
Here, $\mathcal{O}_A^1(n)$ is the coproduct of trees of height ≤ 3 , n leaves in level 2, all of them snaky, and such that all level 3 edges (if any) are straight leaves (the level of an edge is the level of the top vertex). Labels are as above, e.g. ($n = 2$)



The arrows in (2-1) are defined in terms of the following three basic operations with labeled trees, $n, q \geq 0$, $1 \leq i \leq p$,

$$(2-2) \quad \begin{array}{c} \begin{array}{c} \overbrace{A \cdots A}^{n \text{ leaves}} \\ \diagdown \quad \diagup \\ \bullet \\ \mathcal{O}(n) \end{array} \xrightarrow[\text{i.e. } \mathcal{O}\text{-algebra structure map}]{\text{corolla contraction}} \begin{array}{c} A \\ | \\ \bullet \\ \mathcal{O}(1) \end{array} \\ \begin{array}{c} \begin{array}{c} \overbrace{\quad \quad \quad}^q \\ \diagdown \quad \diagup \\ \bullet \\ \mathcal{O}(q) \end{array} \xrightarrow[\text{i.e. operad composition } \circ_i]{\text{inner edge contraction}} \begin{array}{c} \overbrace{\quad \quad \quad}^{p+q-1} \\ \diagdown \quad \diagup \\ \bullet \\ \mathcal{O}(p+q-1) \end{array} \end{array} \end{array} \quad \left| \begin{array}{c} \xrightarrow[\text{i.e. unit } \mathbb{I} \rightarrow \mathcal{O}(1)]{\text{edge subdivision}} \begin{array}{c} \bullet \\ | \\ \bullet \\ \mathcal{O}(1) \end{array} \end{array} \right.$$

Inner edge contraction and edge subdivision also make sense for functor-operads. The two parallel arrows in (2-1) are given by corolla and inner edge contraction, respectively, e.g.



The arrow pointing backwards in (2-1) is given by subdividing straight leaves, e.g.

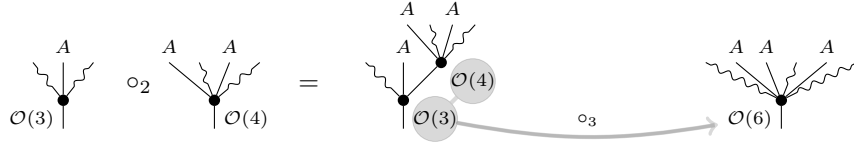


Note that $\mathcal{O}_A(0) = A$, since (2-1) in arity 0 is precisely the final part of the cotriple resolution $\mathcal{F}_{\mathcal{O}}^2(A) \rightrightarrows \mathcal{F}_{\mathcal{O}}(A)$.

The functor-operad structure on \mathcal{O}_A is as follows. Compositions in \mathcal{O}_A are defined as the (reflexive) coequalizer of

$$\begin{array}{ccc} \mathcal{O}_A^1(p) \circ_i \mathcal{O}_A^1(q) & \xrightarrow{\circ_i^1} & \mathcal{O}_A^1(p+q-1) \\ \Downarrow & & \Downarrow \\ \mathcal{O}_A^0(p) \circ_i \mathcal{O}_A^0(q) & \xrightarrow{\circ_i^0} & \mathcal{O}_A^0(p+q-1) \end{array}$$

Here, horizontal maps are defined by contracting the inner edge created by grafting, e.g. \circ_2^0 for $p = q = 2$ contains



The unit is the composite $\text{id}_{\mathcal{C}} \rightarrow \mathcal{O}_A^0(1) \rightarrow \mathcal{O}_A(1)$, where the first arrow is given by edge subdivision

$$\text{id}_{\mathcal{C}} = \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \longrightarrow \begin{array}{c} \text{ } \\ \bullet \\ \text{ } \end{array} \mathcal{O}(1)$$

and inclusion, and the second arrow is the natural projection.

The inclusion of corollas with no straight leaves induces a map $\mathcal{O} \rightarrow \mathcal{O}_A^0$ such that the composite $\mathcal{O} \rightarrow \mathcal{O}_A^0 \rightarrow \mathcal{O}_A$ is a natural morphism of functor-operads.

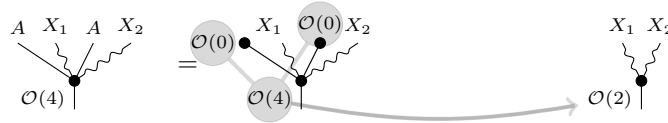
We now compute some enveloping functor-operads. Recall that a *split coequalizer* in a category is a diagram

$$U \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \\ \xrightarrow{t} \end{array} V \begin{array}{c} \xleftarrow{e} \\ \xrightarrow{s} \end{array} W$$

such that $ef = eg$, $es = \text{id}_W$, $ft = \text{id}_V$, and $se = gt$. The arrows pointing \rightarrow are a coequalizer in the usual sense.

Proposition 2.1. *For any operad \mathcal{O} in \mathcal{V} , if $A = z(\mathcal{O}(0))$ is the initial \mathcal{O} -algebra in \mathcal{C} then $\mathcal{O}_A(n)(X_1, \dots, X_n) = z(\mathcal{O}(n)) \otimes \bigotimes_{i=1}^n X_i$, $n \geq 0$.*

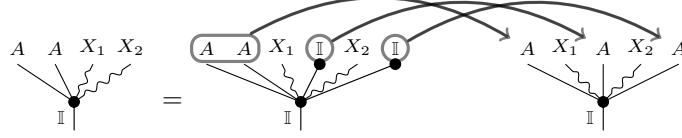
Proof. Since $A = z(\mathcal{O}(0))$ we can cork the straight leaves (a *cork* is an arity 0 inner vertex). The natural projection is then defined by inner edge contraction, e.g. ($n = 2$)



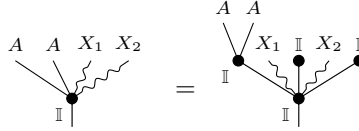
This is a split coequalizer. The two morphisms going backwards are plain inclusions of coproduct factors. \square

Proposition 2.2. *For A a unital associative algebra in \mathcal{C} , $\text{uAss}_A^{\mathcal{V}}(n)(X_1, \dots, X_n) = A \otimes \bigotimes_{i=1}^n (X_i \otimes A)$, $n \geq 0$.*

Proof. The natural projection onto the coequalizer is defined by multiplying consecutive copies of A and introducing an A between consecutive X_i by using the unit $\mathbb{I} \rightarrow A$ (also before X_1 or after X_n if necessary), e.g. ($n = 2$)



This is again a split coequalizer. The two maps going backwards are inclusions of factors, after doing the following kind of identifications,



□

We leave the following similar computation for the reader.

Proposition 2.3. *For any non-unital associative algebra A in \mathcal{C} , we have that $\text{Ass}_A^{\mathcal{V}}(n)(X_1, \dots, X_n) = (A \amalg \mathbb{I}) \otimes \bigotimes_{i=1}^n (X_i \otimes (A \amalg \mathbb{I}))$, $n \geq 1$.*

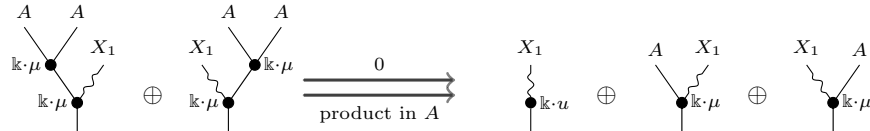
If \mathcal{O} is concentrated in arity 1, we have the following result, which is an obvious consequence of the coequalizer definition.

Proposition 2.4. *If $\mathcal{O}(n) = \emptyset$ for $n \neq 1$, then for any \mathcal{O} -algebra (i.e. left $z(\mathcal{O}(1))$ -module) A in \mathcal{C} , $\mathcal{O}_A(1)(X_1) = z(\mathcal{O}(1)) \otimes X_1$ and $\mathcal{O}_A(n)(X_1, \dots, X_n) = \emptyset$ for $n \geq 2$.*

Notice that $\mathcal{O}_A^1(n)$ contains a corolla with n leaves, all of them snaky. The two maps $\mathcal{O}_A^1(n) \rightrightarrows \mathcal{O}_A^0(n)$ are the identity on this corolla, hence we can neglect it in coequalizer computations. Similarly, if the operad unit is an isomorphism $u: \mathbb{I} \cong \mathcal{O}(1)$ we can neglect the trees in \mathcal{O}_A^1 all whose level 2 inner vertices have arity 1. We use this observation in the proof of the following result.

Proposition 2.5. *In the situation of Example 1.2, for any \mathcal{O} -algebra A , $\mathcal{O}_A(n) = 0$ for $n \geq 3$, $\mathcal{O}_A(2)(X_1, X_2) = X_1 \otimes X_2$, and $\mathcal{O}_A(1)(X_1) = X_1 \oplus (A/A^2) \otimes X_1 \oplus X_1 \otimes (A/A^2)$.*

Proof. The only point which deserves mention is the fact that $\mathcal{O}_A(1)(X_1)$ is by definition the coequalizer of



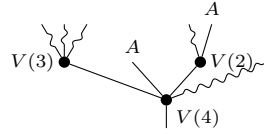
□

We now compute enveloping functor-operads for algebras over free operads.

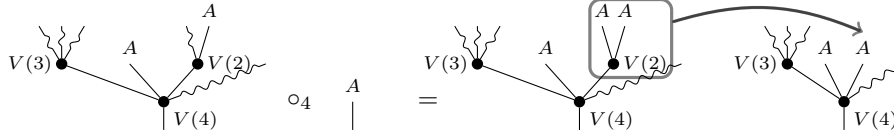
Proposition 2.6. *If $\mathcal{O} = \mathcal{F}(V)$ is the free operad on a sequence $V = \{V(n)\}_{n \geq 0}$ of objects in \mathcal{V} , an \mathcal{O} -algebra in \mathcal{C} is just an object A equipped with structure maps $z(V(n)) \otimes A^{\otimes n} \rightarrow A$, $n \geq 0$, that we regard as corolla contractions,*

$$(2-3) \quad \begin{array}{c} \overbrace{A \cdots A}^{n \text{ leaves}} \\ \swarrow \searrow \\ \bullet \\ | \\ V(n) \end{array} \longrightarrow \begin{array}{c} A \\ | \end{array}$$

and $\mathcal{O}_A(n)$ is the coproduct of all trees equipped with a distinguished subset of n snaky leaves, inner vertices labelled with V and straight leaves with A , which do not contain corollas as in (2-3), e.g.



Composition is simply given by grafting and, if necessary, iterated corolla contractions until no corolla as in (2-3) remains, e.g. $\circ_4: \mathcal{O}_A(5) \circ_4 \mathcal{O}_A(0) \rightarrow \mathcal{O}_A(4)$ contains



The unit is the inclusion in arity 1 of the tree consisting of just one snaky leaf (which represents the identity functor).

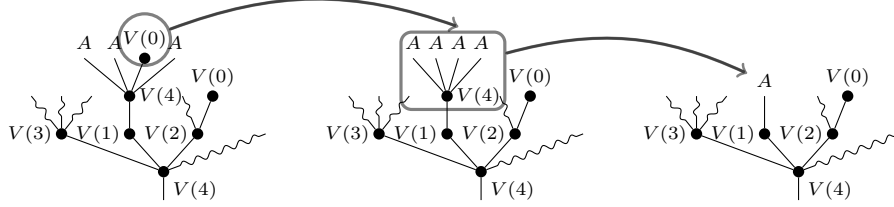
Proof. Recall from [8, §5] the structure of the free operad. The functor $\mathcal{O}_A^0(n)$ is the coproduct of all trees with n snaky leaves where inner vertices are labelled with V and straight leaves with A , e.g. ($n = 5$)

$$(2-4) \quad \begin{array}{c} \begin{array}{c} A \\ | \\ \bullet \\ | \\ V(1) \end{array} \quad \begin{array}{c} V(0) \\ | \\ \bullet \\ | \\ V(2) \end{array} \\ \swarrow \quad \searrow \\ \bullet \\ | \\ V(4) \end{array} \quad \begin{array}{c} V(3) \end{array}$$

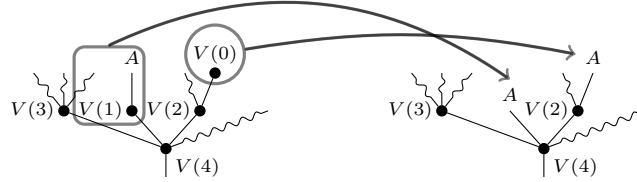
Similarly, $\mathcal{O}_A^1(n)$ is a coproduct indexed by the collections of trees $(T_0; T_1, \dots, T_t)$ such that T_0 has $t + n$ leaves, n of them snaky, and the leaves of T_1, \dots, T_t , if any, are straight. The factor indexed by $(T_0; T_1, \dots, T_t)$ is obtained by labelling inner vertices with V , the leaves of T_1, \dots, T_t with A , and finally grafting each T_i , $1 \leq i \leq t$, into the i^{th} straight leaf of T_0 , e.g. ($t = 1, n = 5$)

$$\left(\begin{array}{c} \begin{array}{c} V(0) \\ | \\ \bullet \\ | \\ V(2) \end{array} \\ \swarrow \quad \searrow \\ \bullet \\ | \\ V(4) \end{array} \quad ; \quad \begin{array}{c} A \quad A \quad V(0) \quad A \\ | \quad | \quad | \quad | \\ \bullet \\ | \\ V(4) \end{array} \right)$$

The maps $\mathcal{O}_A^1(n) \rightrightarrows \mathcal{O}_A^0(n)$ are defined as follows. The first one maps the factor $(T_0; T_1, \dots, T_t)$ to the factor T_0 by iterated corolla contraction $T_i \rightarrow A$, $1 \leq i \leq t$, e.g. starting with the example in the previous diagram



The second one maps via the identity the factor $(T_0; T_1, \dots, T_t)$ to the factor of the tree obtained by grafting each T_i , $1 \leq i \leq t$, into the i^{th} straight leaf of T_0 . The natural projection $\mathcal{O}_A^0(n) \rightarrow \mathcal{O}_A(n)$ is defined by iterated corolla contraction, until no corolla as in (2-3) remains, e.g.



This is a split coequalizer. The arrows going backwards are defined as follows: $\mathcal{O}_A(n) \rightarrow \mathcal{O}_A^0(n)$ is a plain inclusion of coproduct factors, and $\mathcal{O}_A^0(n) \rightarrow \mathcal{O}_A^1(n)$ sends via the identity the factor corresponding to a tree T with n snaky leaves, to the factor $(T_0; T_1, \dots, T_t)$ obtained by pruning T in such a way that the trees with straight leaves T_1, \dots, T_t are as big as possible, e.g. if T is (2-4),

$$\left(\begin{array}{c} \text{Tree with vertices } V(0), V(1), V(2), V(3), V(4) \\ \text{where } V(0) \text{ is a corolla with inputs } A, A, A \text{ and output } V(4) \end{array} ; \begin{array}{c} A \\ | \\ V(1) \end{array}, \begin{array}{c} V(0) \\ | \\ V(1) \end{array} \right)$$

□

The following proposition can be similarly checked, using the explicit construction of the following push-out in $\text{Op}(\mathcal{V})$ [8, §5], where the top map is a free operad map, rather than the structure of free operads,

$$(2-5) \quad \begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(V) \\ g \downarrow & \text{push} & \downarrow g' \\ \mathcal{O} & \xrightarrow{f'} & \mathcal{P} \end{array}$$

A \mathcal{P} -algebra A is the same as an \mathcal{O} -algebra equipped with structure maps $z(V(n)) \otimes A^{\otimes n} \rightarrow A$, that we regard as corolla contractions (2-3) additional to (2-2), fitting

in commutative squares, $n \geq 0$,

$$(2-6) \quad \begin{array}{ccc} U(n) \otimes A^{\otimes n} & \xrightarrow{f(n) \otimes \text{id}} & V(n) \otimes A^{\otimes n} \\ \bar{g}(n) \otimes \text{id} \downarrow & & \downarrow \\ \mathcal{O}(n) \otimes A^{\otimes n} & \xrightarrow{\quad \quad \quad} & A \end{array}$$

Here $\bar{g}: U \rightarrow \mathcal{O}$ in $\mathcal{V}^{\mathbb{N}}$ is the adjoint of g in (2-5).

Any map of operads $\phi: \mathcal{O} \rightarrow \mathcal{P}$ and \mathcal{P} -algebra A give rise to a natural map of functor-operads $\phi_A: \mathcal{O}_A \rightarrow \mathcal{P}_A$ induced by the map of diagrams $\phi_A^i: \mathcal{O}_A^i \rightarrow \mathcal{P}_A^i$ defined by ϕ on inner vertices, $i = 0, 1$.

Proposition 2.7. *Given an operad push-out (2-5) and a \mathcal{P} -algebra A , $f'_A: \mathcal{O}_A \rightarrow \mathcal{P}_A$ is the transfinite composition of a sequence*

$$\mathcal{O}_A = \mathcal{P}_{A,0} \xrightarrow{\Phi_1} \mathcal{P}_{A,1} \rightarrow \cdots \rightarrow \mathcal{P}_{A,t-1} \xrightarrow{\Phi_t} \mathcal{P}_{A,t} \rightarrow \cdots$$

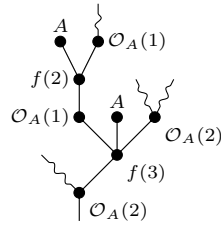
in $\mathcal{C}^{\mathbb{N}}$ such that Φ_t , $t \geq 1$, is given by the push-out square

$$\begin{array}{ccc} \bullet & \xrightarrow{\tilde{\Phi}_t} & \bullet \\ \Psi_t \downarrow & \text{push} & \downarrow \bar{\Psi}_t \\ \mathcal{P}_{A,t-1} & \xrightarrow{\Phi_t} & \mathcal{P}_{A,t} \end{array}$$

where $\tilde{\Phi}_t(n)$ is the coproduct of all trees with n leaves in even level, all of them snaky, no leaves in odd level, t even inner vertices, labelled with f , odd inner vertices labelled with \mathcal{O}_A , and such that T does not contain

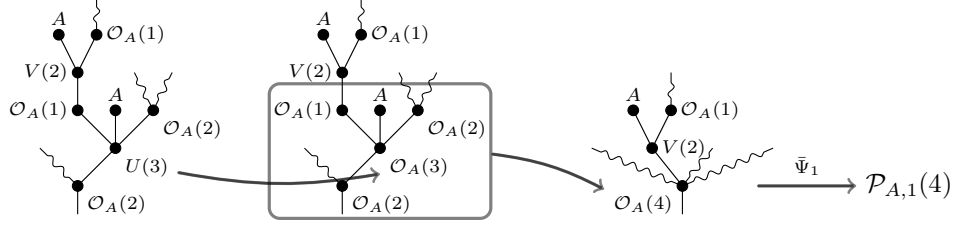
$$(2-7) \quad \begin{array}{c} \overbrace{A \cdots A}^{n \text{ leaves}} \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \downarrow \\ f(n) \end{array}$$

where $n \geq 0$, e.g. ($t = 2, n = 4$)



The attaching map Ψ_t is defined as the composition of three maps from the same trees, where now one even inner vertex is labelled with U and the rest with V . The first map is induced by the composition of $\bar{g}: U \rightarrow \mathcal{O}$ and the natural map $\mathcal{O} \rightarrow \mathcal{O}_A$, the second map is defined by inner edge contraction, and the third map

is the restriction of $\bar{\Psi}_{t-1}$ if $t > 1$ or the identity if $t = 1$, e.g.



The only remarkable cosmetic difference between Propositions 2.6 and 2.7 is that, in the former, we allow straight leaves but not corks (corks are corollas (2-3) for $n = 0$) and, in the latter, we allow corks in odd levels (not in even levels, since (2-7) is an even cork for $n = 0$) but not straight leaves at all. This is just to simplify notation. We do not describe the functor-operad structure of \mathcal{P}_A in Proposition 2.7. It is given by grafting, then contracting the new inner edge and, if they appear, contracting corollas (2-7) by using the \mathcal{P} -algebra structure maps $z(V(n)) \otimes A^{\otimes n} \rightarrow A$. To the best of our knowledge, Proposition 2.7 is new even in the symmetric case. In that case, enveloping functor-operads can be replaced with honest enveloping operads in the obvious way.

An \mathcal{O} -algebra map $\phi: A \rightarrow B$ induces a functor-operad map $\mathcal{O}_\phi: \mathcal{O}_A \rightarrow \mathcal{O}_B$ which is the coequalizer of maps $\mathcal{O}_\phi^i: \mathcal{O}_A^i \rightarrow \mathcal{O}_B^i$, $i = 0, 1$, induced by ϕ on straight leaves.

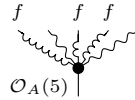
Proposition 2.8. *If we have an \mathcal{O} -algebra push-out (1-1), $\mathcal{O}_{f'}$ is the transfinite composition of a sequence*

$$\mathcal{O}_A = \mathcal{O}_{B,0} \xrightarrow{\Phi_1} \mathcal{O}_{B,1} \rightarrow \cdots \rightarrow \mathcal{O}_{B,t-1} \xrightarrow{\Phi_t} \mathcal{O}_{B,t} \rightarrow \cdots$$

in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ such that Φ_t , $t \geq 1$, is given by the push-out square

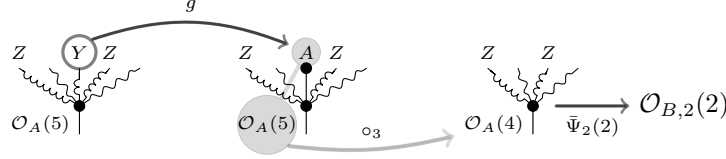
$$\begin{array}{ccc} \bullet & \xrightarrow{\tilde{\Phi}_t} & \bullet \\ \Psi_t \downarrow & \text{push} & \downarrow \bar{\Psi}_t \\ \mathcal{O}_{B,t-1} & \xrightarrow{\Phi_t} & \mathcal{O}_{B,t} \end{array}$$

where $\tilde{\Phi}_t(n)$ is the coproduct of all corollas with $t+n$ leaves, t bumpy and n snaky, inner vertex labelled with \mathcal{O}_A , and bumpy leaves with f , e.g. ($t = 3, n = 2$)

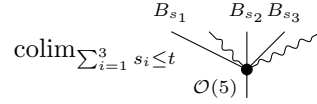


The attaching map Ψ_t is defined as the composition of three maps from the same corollas, where now one bumpy leaf is labelled with Y and the rest with Z . The first map is given by $\bar{g}: Y \rightarrow A = \mathcal{O}_A(0)$ (which straightens and corks one bumpy leaf), the second map is defined by inner edge contraction, and the third map is the

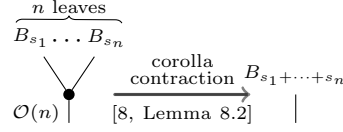
restriction of $\bar{\Psi}_{t-1}$ if $t > 1$ and the identity if $t = 1$, e.g.



Proof. By [8, §8] (corrected version), we can define $\mathcal{O}_{B,t}$ as the coequalizer of the maps $\mathcal{O}_{B,t}^1 \rightrightarrows \mathcal{O}_{B,t}^0$ defined as follows. The functor $\mathcal{O}_{B,t}^i(n)$ is a coproduct with the same indexing set as $\mathcal{O}_A^i(n)$. The factor corresponding to a tree with n snaky leaves is the colimit of the functors obtained by labeling the inner vertices with \mathcal{O} and the straight leaves with B_{s_1}, \dots, B_{s_r} , $s_1 + \dots + s_r \leq t$, e.g. ($i = 0, n = 2, r = 3$)



The maps $\mathcal{O}_{B,t}^1 \rightrightarrows \mathcal{O}_{B,t}^0$ are respectively given by inner edge contraction, as above, and by the following new kind of corolla contraction,



The map Φ_t is the coequalizer of two maps $\Phi_t^1 \rightrightarrows \Phi_t^0$ in $\text{Mor}(\mathcal{C})^{\mathcal{C}^{(\mathbb{N})}}$,

$$(2-8) \quad \begin{array}{ccc} \mathcal{O}_{B,t-1}^1 & \xrightarrow{\Phi_t^1} & \mathcal{O}_{B,t}^1 \\ \Downarrow & & \Downarrow \\ \mathcal{O}_{B,t-1}^0 & \xrightarrow{\Phi_t^0} & \mathcal{O}_{B,t}^0 \end{array}$$

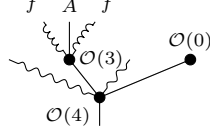
where Φ_t^i is defined by the bonding maps in [8, Lemma 8.1] (corrected version).

The map Φ_t^i in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ fits into a push-out diagram

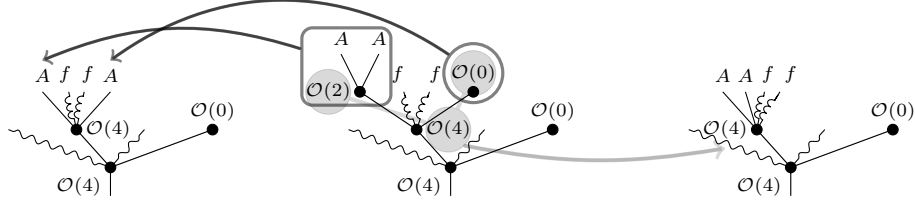
$$(2-9) \quad \begin{array}{ccc} \bullet & \xrightarrow{\bar{\Phi}_t^i} & \bullet \\ \downarrow & \text{push} & \downarrow \\ \mathcal{O}_{B,t-1}^i & \xrightarrow{\Phi_t^i} & \mathcal{O}_{B,t}^i \end{array}$$

where $\bar{\Phi}_t^i$ is the coequalizer of the pair $\bar{\Phi}_t^{i1} \rightrightarrows \bar{\Phi}_t^{i0}$ in $\text{Mor}(\mathcal{C})^{\mathcal{C}^{(\mathbb{N})}}$ defined as follows. For $i = 0$, $\bar{\Phi}_t^{00}(n)$ is a coproduct of trees of height 3 with n leaves in level 2, all of them snaky, and such that all level 3 edges are leaves, including t bumpy leaves (the rest straight). Inner edges are labeled with \mathcal{O} , straight leaves with A , and bumpy

leaves with f , e.g. $(n = 2, t = 2)$

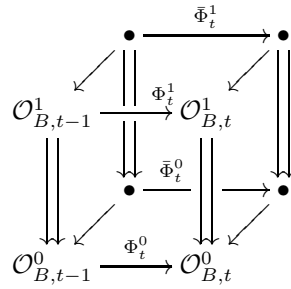


Similarly, for $i = 1$, $\bar{\Phi}_t^{01}(n)$ is a coproduct of trees of height ≤ 4 with n level 2 leaves (snaky), t level 3 leaves (bumpy), and such that all level 4 edges (if any) are straight leaves. Labels are as above, and the maps $\bar{\Phi}_t^{01} \rightrightarrows \bar{\Phi}_t^{00}$ are defined by corolla contraction and level 3 inner edge contraction, e.g. $(n = 2, t = 2)$



Analogously, $\bar{\Phi}_r^{10}(n)$ is the coproduct of height 4 trees with n level 2 leaves (snaky), no level 3 leaves, and such that all level 4 edges are leaves (t of them bumpy), $\bar{\Phi}_r^{11}(n)$ is the coproduct of trees of height ≤ 5 with n level 2 leaves (snaky), no level 3 leaves, t level 4 leaves (bumpy), and such that all level 5 edges (if any) are straight leaves. Labels are again as above, and the maps $\bar{\Phi}_t^{11} \rightrightarrows \bar{\Phi}_t^{10}$ are defined by corolla contraction and level 4 inner edge contraction, respectively. The attaching maps in (2-9) are derived from the attaching maps in [8, Lemma 8.1] (corrected version).

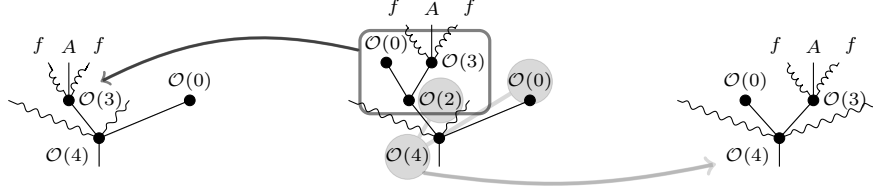
There are parallel maps $\bar{\Phi}_t^1 \rightrightarrows \bar{\Phi}_t^0$ in $\text{Mor}(\mathcal{C})^{\mathcal{C}^{(N)}}$ compatible with $\Phi_t^1 \rightrightarrows \Phi_t^0$ in (2-8) via the attaching and characteristic maps in (2-9),



The maps $\bar{\Phi}_t^1 \rightrightarrows \bar{\Phi}_t^0$ are obtained by taking horizontal coequalizers in the following diagram,

$$(2-10) \quad \begin{array}{ccc} \bar{\Phi}_t^{11} & \rightrightarrows & \bar{\Phi}_t^{10} \\ \Downarrow & & \Downarrow \\ \bar{\Phi}_t^{01} & \rightrightarrows & \bar{\Phi}_t^{00} \end{array}$$

Here, the right (resp. left) vertical arrow in each vertical pair is defined by contraction of level 2 (resp. 3) inner edges, e.g. the vertical pair on the right,



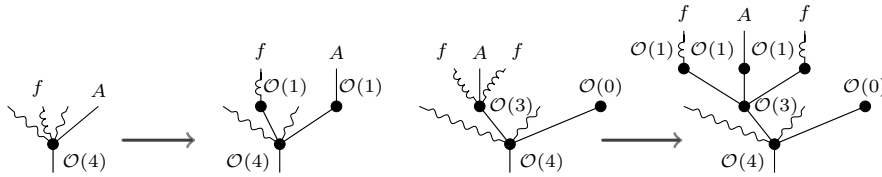
Hence Φ_t is the push-out of $\bar{\Phi}_t$, the coequalizer of $\bar{\Phi}_t^1 \rightrightarrows \bar{\Phi}_t^0$,

$$\begin{array}{ccc} \bullet & \xrightarrow{\bar{\Phi}_t} & \bullet \\ \downarrow & \text{push} & \downarrow \\ \mathcal{O}_{B,t-1} & \xrightarrow{\Phi_t} & \mathcal{O}_{B,t} \end{array}$$

The map $\bar{\Phi}_t$ can also be obtained by taking first vertical and then horizontal coequalizers in (2-10). Let $\tilde{\Phi}_t^j$ be the coequalizer of $\bar{\Phi}_t^{1j} \rightrightarrows \bar{\Phi}_t^{0j}$. The functor $\tilde{\Phi}_t^0(n)$ is the coproduct of all corollas with t bumpy leaves, n snaky leaves, and an undetermined number of straight leaves, with labels as above. The natural projection is defined by inner edge contraction, e.g.



This is indeed a split coequalizer. The arrows going backwards are defined by subdivision of straight and bumpy leaves, e.g.



One can similarly check that $\tilde{\Phi}_t^1(n)$ is the coproduct of all trees of height ≤ 3 with $t + n$ leaves at level 2, t bumpy and n snaky, such that all level 3 edges (if any) are straight leaves, with the usual labeling. Moreover, the parallel arrows $\tilde{\Phi}_t^1 \rightrightarrows \tilde{\Phi}_t^0$ obtained by taking vertical coequalizers in (2-10) are defined by inner edge contraction and corolla contraction, respectively. Therefore $\bar{\Phi}_t$, regarded as the coequalizer of $\bar{\Phi}_t^1 \rightrightarrows \bar{\Phi}_t^0$, coincides with the arrow $\bar{\Phi}_t$ in the statement. \square

Unlike Proposition 2.6, Proposition 2.8 was previously known in the symmetric case, at least implicitly. It follows from the proof of [2, Proposition 5.4] and from the construction of push-outs of free operad maps in [8, §5].

3. PROOFS

We start with a generalization of the Reedy model structure on the category of diagrams indexed by $\mathbf{2}^n$, see [6, §5.1] and [5, §15.3].

Proposition 3.1. *If \mathcal{M} is a model category and $S \subset \{1, \dots, n\}$, there is a model structure $\mathcal{M}_S^{\mathbf{2}^n}$ on the diagram category $\mathcal{M}^{\mathbf{2}^n}$ such that a map $\tau: F \rightarrow G$ is*

- *a fibration if $\tau(x_1, \dots, x_n): F(x_1, \dots, x_n) \rightarrow G(x_1, \dots, x_n)$ is a fibration in \mathcal{M} for all $(x_1, \dots, x_n) \in \mathbf{2}^n$,*
- *a weak equivalence if $\tau(x_1, \dots, x_n)$ is a weak equivalence in \mathcal{M} for all $(x_1, \dots, x_n) \in \mathbf{2}^n$ with $x_i = 0$ if $i \in S$,*
- *and a cofibration if the relative latching map of τ at any $(x_1, \dots, x_n) \in \mathbf{2}^n$ is a cofibration, and moreover a trivial cofibration if $x_i = 1$ for some $i \in S$.*

Proof. For $n = 0$ and $S = \emptyset$ (the only choice) we recover the given model structure on $\mathcal{M} = \mathcal{M}^{\mathbf{2}^0}$. For $n = 1$, $\mathcal{M}_{\emptyset}^{\mathbf{2}}$ is the Reedy model structure. Let us check that $\mathcal{M}_{\{1\}}^{\mathbf{2}}$ satisfies the axioms of model categories [6, Definition 1.1.3]. Only the parts of the factorization and lifting axioms involving a cofibration and a trivial fibration are not completely trivial.

A map $X \rightarrow Y$ in $\mathcal{M}_{\{1\}}^{\mathbf{2}}$ can be factored as $X \rightarrow Z \xrightarrow{\sim} Y$ in the following way. The map $X \rightarrow Y$ is the outer square in the following commutative diagram,

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{\quad} & Z_0 & \xrightarrow{\sim} & Y_0 \\
 \downarrow & \text{push} & \swarrow & \downarrow & \downarrow \\
 X_1 & \xrightarrow{\quad} & P & \xrightarrow{\sim} & Z_1 & \xrightarrow{\quad} & Y_1
 \end{array}$$

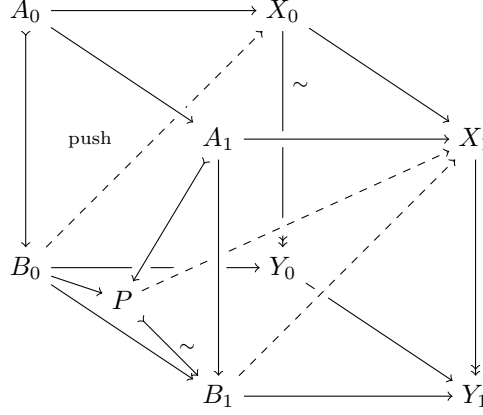
(Note: The diagram above is a simplified representation of the commutative diagram in the image. The image shows a more complex diagram with curved arrows and a central square.)

Here, we first factor $X_0 \rightarrow Y_0$ as a cofibration followed by a trivial fibration in \mathcal{M} . Then we factor the induced map from the push-out $P = Z_0 \cup_{X_0} X_1 \rightarrow Y_1$ as a trivial cofibration followed by a fibration.

A diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow \sim \\
 B & \xrightarrow{\quad} & Y
 \end{array}$$

in $\mathcal{M}_{\{1\}}^2$ is the same as a commutative cube



We can construct a lifting $B \rightarrow X$ as follows. We first take a lifting $B_0 \rightarrow X_0$ in \mathcal{M} of the rear commutative square. This lifting and the universal property of a push-out induce a map $P = B_0 \cup_{A_0} A_1 \rightarrow X_1$, which is the top arrow of a commutative square containing also $B_1 \rightarrow Y_1$. We can take a lifting $B_1 \rightarrow X_1$ of this square in \mathcal{M} too. These two liftings in \mathcal{M} define the lifting in $\mathcal{M}_{\{1\}}^2$.

For $n > 1$, using the exponential law $\mathcal{M}^{2^n} = \mathcal{M}^{2^{n-1} \times 2} = (\mathcal{M}^{2^{n-1}})^2$, it is easy to see that $\mathcal{M}_S^{2^n} = (\mathcal{M}_S^{2^{n-1}})^2_\emptyset$ if $n \notin S$ and $\mathcal{M}_S^{2^n} = (\mathcal{M}_{S \setminus \{n\}}^{2^{n-1}})^2_{\{1\}}$ if $n \in S$, so the proposition follows by induction. \square

Remark 3.2. Notice that fibrations in $\mathcal{M}_S^{2^n}$ are independent of S , they are the same as in the Reedy model structure $\mathcal{M}_\emptyset^{2^n}$, hence the same holds for trivial cofibrations (they are the maps whose relative latching maps are trivial cofibrations in \mathcal{M}). This means that $\mathcal{M}_S^{2^n}$ is a right Bousfield localization of $\mathcal{M}_\emptyset^{2^n}$.

We remind the reader that cofibrant objects in $\mathcal{M}_\emptyset^{2^n}$ are functors F whose latching maps are cofibrations. They take values in cofibrant objects in \mathcal{M} and have cofibrant latching objects. Moreover, any weak equivalence between cofibrant functors induces weak equivalences between latching objects.

Given model categories \mathcal{M} and \mathcal{N} , we introduce some naive homotopical notions in big functor categories.

Definition 3.3. A map $\tau: F \rightarrow G$ in $\mathcal{M}^{\mathcal{N}^n}$ is a weak equivalence, fibration or cofibration if, given cofibrations between cofibrant objects g_1, \dots, g_n in \mathcal{N} , $\tau(g_1, \dots, g_n)$ is so in $\mathcal{M}_S^{2^n}$ for any $S \subset \{1, \dots, n\}$ such that g_i is a trivial cofibration if $i \in S$. A map in $\mathcal{M}^{\mathcal{N}^{(n)}}$ is a weak equivalence, fibration or cofibration if it is so aritywise. Cofibrant objects are defined in the usual way.

Remark 3.4. Notice that $S = \emptyset$ is always the smallest subset satisfying the assumptions in the previous definition. For this choice, we obtain the strongest conditions on weak equivalences. Indeed, $\tau: F \rightarrow G$ is a weak equivalence in $\mathcal{M}^{\mathcal{N}^n}$ if and only if $\tau(X_1, \dots, X_n): F(X_1, \dots, X_n) \rightarrow G(X_1, \dots, X_n)$ is a weak equivalence in \mathcal{M} for any cofibrant objects X_1, \dots, X_n in \mathcal{N} . This choice also says that cofibrations in $\mathcal{M}^{\mathcal{N}^n}$ yield Reedy cofibrations when evaluated at cofibrations between cofibrant objects in \mathcal{N} , but they must satisfy extra conditions obtained for

the biggest choice of S . Similarly for cofibrant functors, which preserve cofibrant objects. Notice that for $n = 0$ we recover the original notions in $\mathcal{M}^{\mathcal{N}^0} = \mathcal{M}$.

The identity functor is cofibrant in $\mathcal{M}^{\mathcal{M}}$. Suppose that \mathcal{M} is a monoidal model category. The n -fold tensor product is cofibrant in $\mathcal{M}^{\mathcal{M}^n}$ by the push-out product axiom, $n \geq 2$. More generally, if Y is a cofibrant object in \mathcal{M} , the functor $(X_1, \dots, X_n) \mapsto Y \otimes \bigotimes_{i=1}^n X_i$ is cofibrant in $\mathcal{M}^{\mathcal{M}^n}$, and if f is a weak equivalence between cofibrant objects or a (trivial) cofibration in \mathcal{M} then so is the natural transformation $(X_1, \dots, X_n) \mapsto f \otimes \bigotimes_{i=1}^n X_i$ in $\mathcal{M}^{\mathcal{M}^n}$, $n \geq 0$.

Proposition 3.5. *Let \mathcal{O} be an operad in \mathcal{V} such that $\mathcal{O}(n)$ is cofibrant in \mathcal{C} for all $n \geq 0$. Then for any cofibrant A in $\text{Alg}_{\mathcal{C}}(\mathcal{O})$, \mathcal{O}_A is cofibrant in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$. Moreover, for any cofibration with cofibrant source $f': A \rightarrow B$ in $\text{Alg}_{\mathcal{C}}(\mathcal{O})$, $\mathcal{O}_{f'}: \mathcal{O}_A \rightarrow \mathcal{O}_B$ is a cofibration in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$.*

Proof. Generating cofibrations in $\text{Alg}_{\mathcal{C}}(\mathcal{O})$ are free \mathcal{O} -algebra maps on cofibrations in \mathcal{C} , see [8, §9]. Hence, by the usual argument involving transfinite compositions and retracts, it is enough to notice that the first statement holds for the initial \mathcal{O} -algebra $A = z(\mathcal{O}(0))$ in \mathcal{C} , see Proposition 2.1 and Remark 3.4, and then check the second statement when f' fits into a push-out (1-1) with f a cofibration in \mathcal{C} , assuming that the first statement holds for A . Let us check this.

We can replace f with the bottom map in the following push-out in \mathcal{C} ,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \bar{g} \downarrow & \text{push} & \downarrow \\ A & \longrightarrow & A \cup_X Y \end{array}$$

Compare the trick at the beginning of the proof of [9, Proposition 4.2]. Therefore, since $A = \mathcal{O}_A(0)$ is cofibrant in \mathcal{C} , we can suppose that X , and hence Y , is cofibrant. The factors of the coproduct $\tilde{\Phi}_t(n)$ in Proposition 2.8 are cofibrations with cofibrant source and target in $\mathcal{C}^{\mathcal{C}^n}$ for all $n \geq 0$ and $t \geq 1$. This follows from the facts that f is a cofibration between cofibrant objects and $\mathcal{O}_A(n)$ is cofibrant in $\mathcal{C}^{\mathcal{C}^n}$ for $n \geq 1$. Hence Φ_t is a cofibration for all $t \geq 1$, and $\mathcal{O}_{f'}$ is a cofibration since it is a transfinite composition of cofibrations. This also proves that $\mathcal{O}_{B,t}$ is cofibrant in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$, $t \geq 0$. This fact will be used later. \square

Proposition 3.6. *Let $\phi: \mathcal{O} \xrightarrow{\sim} \mathcal{P}$ be a weak equivalence in $\text{Op}(\mathcal{V})$. Assume that the objects $\mathcal{O}(n)$ and $\mathcal{P}(n)$ are cofibrant in \mathcal{V} for all $n \geq 0$. Given a cofibrant \mathcal{O} -algebra A in \mathcal{C} , the map $\phi_{\eta_A}: \mathcal{O}_A \rightarrow \mathcal{P}_{\phi_* A}$ induced by ϕ and by the unit $\eta_A: A \rightarrow \phi^* \phi_* A$ of the change of operad adjunction $\phi_* \dashv \phi^*$ [8, (1)] is a weak equivalence in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$.*

Proof. As in the proof of Proposition 3.5, it is enough to check the statement for $A = z(\mathcal{O}(0))$ the initial \mathcal{O} -algebra, and then for B in (1-1) assuming that it holds for A and that f is a cofibration between cofibrant objects.

For the initial \mathcal{O} -algebra, ϕ_{η_A} is induced by the sequence of weak equivalences between cofibrant objects $z(\phi)$, see Proposition 2.1 and Remark 3.4, so it is a weak equivalence.

In the situation of (1-1), we have another push-out square

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{P}}(Y) & \xrightarrow{\mathcal{F}_{\mathcal{P}}(f)} & \mathcal{F}_{\mathcal{P}}(Z) \\ \phi_* g \downarrow & \text{push} & \downarrow \phi_* g' \\ \phi_* A & \xrightarrow{\phi_* f'} & \phi_* B \end{array}$$

where $\phi_* g$ and $\phi_* g'$ are adjoints to \bar{g} and \bar{g}' , respectively. Notice that $\phi_* A$ is a cofibrant \mathcal{P} -algebra, since ϕ_* is a left Quillen functor. Using Proposition 2.8, ϕ_{η_B} is the colimit of a sequence

$$\begin{array}{ccccccc} \mathcal{O}_A = \mathcal{O}_{B,0} & \xrightarrow{\Phi_1^{\mathcal{O}}} & \mathcal{O}_{B,1} & \rightarrow \cdots \rightarrow & \mathcal{O}_{B,t-1} & \xrightarrow{\Phi_t^{\mathcal{O}}} & \mathcal{O}_{B,t} \rightarrow \cdots \\ \phi_{\eta_A} \downarrow \sim & & \downarrow \phi_{\eta_{B,0}} & & \downarrow \phi_{\eta_{B,1}} & & \downarrow \phi_{\eta_{B,t-1}} & & \downarrow \phi_{\eta_{B,t}} \\ \mathcal{P}_{\phi_* A} = \mathcal{P}_{\phi_* B,0} & \xrightarrow{\Phi_1^{\mathcal{P}}} & \mathcal{P}_{\phi_* B,1} & \rightarrow \cdots \rightarrow & \mathcal{P}_{\phi_* B,t-1} & \xrightarrow{\Phi_t^{\mathcal{P}}} & \mathcal{P}_{\phi_* B,t} \rightarrow \cdots \end{array}$$

where all objects are cofibrant and horizontal maps are cofibrations, see Proposition 3.5 and its proof. The map $\phi_{\eta_{B,t}}$ is obtained by taking horizontal push-outs in the following commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{B,t-1} & \xleftarrow{\Psi_t^{\mathcal{O}}} & \bullet & \xleftarrow{\tilde{\Phi}_t^{\mathcal{O}}} & \bullet \\ \phi_{\eta_{B,t-1}} \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_{\phi_* B,t-1} & \xleftarrow{\Psi_t^{\mathcal{P}}} & \bullet & \xleftarrow{\tilde{\Phi}_t^{\mathcal{P}}} & \bullet \end{array}$$

All objects in this diagram are cofibrant and the two arrows pointing \rightarrow are cofibrations, see the proof of Proposition 3.5 again. The square on the right is aritwise the factor-preserving map between coproducts $\tilde{\Phi}_t^{\mathcal{O}}(n) \rightarrow \tilde{\Phi}_t^{\mathcal{P}}(n)$ in $\text{Mor}(\mathcal{C})^{\mathcal{C}^n}$ induced by ϕ_{η_A} on inner vertices, e.g. ($n = 2$)

This map is a weak equivalence since f is a cofibration between cofibrant objects and ϕ_{η_A} is a weak equivalence (by assumption) between cofibrant objects (by Proposition 3.5). The starting map $\phi_{\eta_{B,0}} = \phi_{\eta_A}$ is a weak equivalence, hence we deduce by induction, using the cube lemma [6, Lemma 5.2.6], that the maps $\phi_{\eta_{B,t}}$ are weak equivalences for all $t \geq 0$. Since ϕ_{η_B} is the colimit of weak equivalences between sequences of cofibrations with cofibrant starting objects, we deduce that ϕ_{η_B} is also a weak equivalence [5, Proposition 15.10.12 (1)]. Notice that [5, Proposition 15.10.12 (1)] is also true for continuous sequences indexed by an arbitrary ordinal, compare [5, Corollary 5.1.6]. This generalization is used to check the limit steps in the transfinite induction. \square

Let $L: \mathcal{M} \rightleftarrows \mathcal{N}: R$ be a Quillen pair. Left composition with L induces a ‘functor’ $\mathcal{M}^{\mathcal{M}^n} \rightarrow \mathcal{N}^{\mathcal{M}^n}$ which preserves (trivial) cofibrations, $n \geq 0$. Right composition with $L^{\times n} = L \times \cdots \times L$ gives rise to a ‘functor’ $\mathcal{N}^{\mathcal{N}^n} \rightarrow \mathcal{N}^{\mathcal{M}^n}$

which preserves all homotopical notions in Definition 3.3, $n \geq 0$. We will denote $L^{\times(\mathbb{N})} = \{L^{\times n}\}_{n \geq 0}$.

Let us place ourselves in of [9, §7]. There is a natural map in $\mathcal{D}^{\mathcal{C}^{(\mathbb{N})}}$,

$$\chi_{\mathcal{O},A}: \bar{F}\mathcal{O}_A \longrightarrow F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(A)} \bar{F}^{\times(\mathbb{N})}$$

defined by taking vertical coequalizers in the following diagram

$$\begin{array}{ccc} \bar{F}\mathcal{O}_A^1 & \xrightarrow{\chi_{\mathcal{O},A}^1} & F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(A)}^1 \bar{F}^{\times(\mathbb{N})} \\ \Downarrow & & \Downarrow \\ \bar{F}\mathcal{O}_A^0 & \xrightarrow{\chi_{\mathcal{O},A}^0} & F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(A)}^0 \bar{F}^{\times(\mathbb{N})} \end{array}$$

The functor \bar{F} preserves coproducts since it is a left adjoint, and $\chi_{\mathcal{O},A}^0(n)$ is factor-wise defined as illustrated in the following picture ($n = 2$)

$$\bar{F} \left(\begin{array}{c} A \quad A \quad A \\ \swarrow \quad \downarrow \quad \searrow \\ \mathcal{O}(5) \end{array} \right) \longrightarrow \begin{array}{c} \bar{F}(A) \quad \bar{F}(-) \quad \bar{F}(A) \quad \bar{F}(A) \quad \bar{F}(-) \\ \swarrow \quad \downarrow \quad \searrow \\ \boxed{F(\mathcal{O}(5))} \end{array} \longrightarrow \begin{array}{c} \bar{F}(A) \quad \bar{F}(-) \quad \bar{F}(A) \quad \bar{F}(A) \quad \bar{F}(-) \\ \swarrow \quad \downarrow \quad \searrow \\ F^{\text{oper}}(\mathcal{O}(5)) \end{array}$$

Here, the first arrow is defined by the comultiplication of \bar{F} and by the natural transformation $\tau: \bar{F}z_{\mathcal{C}} \rightarrow z_{\mathcal{D}}F$, and the second one is defined by the map $\chi: F(\mathcal{O}) \rightarrow F^{\text{oper}}(\mathcal{O})$ of sequences in \mathcal{V} [9, (4-1)]. Similarly $\chi_{\mathcal{O},A}^1(n)$.

Proposition 3.7. *If $\bar{F} \dashv \bar{G}$ is a weak monoidal Quillen adjunction, \mathcal{V} and \mathcal{W} have cofibrant tensor units, \mathcal{O} is a cofibrant operad in \mathcal{V} , and A is a cofibrant \mathcal{O} -algebra in \mathcal{C} , then $\chi_{\mathcal{O},A}$ is a weak equivalence in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$.*

Proof. It is enough to prove the statement just in the two cases described at the beginning of the poof of Proposition 3.6.

If A is the initial \mathcal{O} -algebra, then $\bar{F}\mathcal{O}(A)$ is the initial $F^{\text{oper}}(\mathcal{O})$ -algebra and $\chi_{\mathcal{O},A}$ is defined like $\chi_{\mathcal{O},A}^0$ above on corollas without straight leaves. Comultiplication is a weak equivalence when evaluated at cofibrant objects, since \bar{F} is a weak monoidal left Quillen functor [9, Definition 2.6]. The natural transformation τ is also a weak equivalence between cofibrant objects when evaluated at $\mathcal{O}(n)$, since it is cofibrant in \mathcal{V} [9, Corollary 3.8]. Moreover, $\chi_{\mathcal{O}}$ is a sequence of weak equivalences between cofibrant objects by [9, Proposition 4.2]. This shows that $\chi_{\mathcal{O},A}$ is a weak equivalence in this case.

Assume we have a push-out (1-1) such that f is a cofibration between cofibrant objects and A satisfies the proposition. Then $\chi_{\mathcal{O},B}$ is the colimit of a sequence where all objects are cofibrant and horizontal maps are cofibrations, see Proposition 3.5 and its proof,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bar{F}\mathcal{O}_{B,t-1} & \xrightarrow{\bar{F}\Phi_t^{\mathcal{O}}} & \bar{F}\mathcal{O}_{B,t} & \longrightarrow & \cdots \\ & & \downarrow \chi_{\mathcal{O},B,t-1} & & \downarrow \chi_{\mathcal{O},B,t} & & \\ \cdots & \longrightarrow & F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(B),t-1} \bar{F}^{\times(\mathbb{N})} & \xrightarrow{\Phi_t^{F^{\text{oper}}(\mathcal{O})} \bar{F}^{\times(\mathbb{N})}} & F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(B),t} \bar{F}^{\times(\mathbb{N})} & \longrightarrow & \cdots \end{array}$$

This sequence starts with the weak equivalence $\chi_{\mathcal{O},B,0} = \chi_{\mathcal{O},A}$. The map $\chi_{\mathcal{O},B,t}$ is obtained by taking horizontal push-outs in the following commutative diagram

$$\begin{array}{ccccc}
 \bar{F}\mathcal{O}_{B,t-1} & \xleftarrow{\bar{F}\Psi_t^{\mathcal{O}}} & \bullet & \xrightarrow{\bar{F}\Phi_t^{\mathcal{O}}} & \bullet \\
 \downarrow \phi_{\eta_{B,t-1}} & & \downarrow & & \downarrow \\
 F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(B),t-1} \bar{F}^{\times}(\mathbb{N}) & \xleftarrow{\Psi_t^{F^{\text{oper}}(\mathcal{O})} \bar{F}^{\times}(\mathbb{N})} & \bullet & \xrightarrow{\tilde{\Phi}_t^{F^{\text{oper}}(\mathcal{O})} \bar{F}^{\times}(\mathbb{N})} & \bullet
 \end{array}$$

The two arrows pointing \rightarrow are cofibrations between cofibrant objects, see again the proof of Proposition 3.5. The square on the right is aritywise the factor-preserving map between coproducts in $\text{Mor}(\mathcal{C})^{\mathcal{C}^n}$ defined by $\chi_{\mathcal{O},A}$ on inner vertices, e.g. ($n = 2$)

$$\begin{array}{c}
 \begin{array}{c} f \quad f \quad f \\ \diagdown \quad | \quad \diagup \\ \bullet \\ \text{---} F\mathcal{O}_A(5) \end{array} \xrightarrow{\chi_{\mathcal{O},A}(5)} \begin{array}{c} f \quad f \quad f \\ \diagdown \quad | \quad \diagup \\ \bullet \\ \text{---} F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(A)}(5) \bar{F}^{\times 5} \end{array} = \begin{array}{c} \bar{F}(f) \quad \bar{F}(-) \quad \bar{F}(f) \quad \bar{F}(f) \quad \bar{F}(-) \\ \diagdown \quad | \quad \diagup \quad | \quad \diagdown \\ \bullet \\ \text{---} F^{\text{oper}}(\mathcal{O})_{\bar{F}\mathcal{O}(A)}(5) \end{array}
 \end{array}$$

Since A satisfies the proposition, this is a weak equivalence, and the result follows as in the last paragraph of the proof of Proposition 3.6. \square

We now abandon the context of [9, §7].

Definition 3.8. An operad \mathcal{O} in \mathcal{V} is *excellent* if:

- (1) Given an \mathcal{O} -algebra A in \mathcal{C} with underlying cofibrant object, \mathcal{O}_A is cofibrant in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$.
- (2) For any weak equivalence $\varphi: A \rightarrow C$ between \mathcal{O} -algebras in \mathcal{C} with underlying cofibrant objects, $\mathcal{O}_{\varphi}: \mathcal{O}_A \rightarrow \mathcal{O}_C$ is a weak equivalence in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$.

Proposition 3.9. *The initial operad, $\text{Ass}^{\mathcal{V}}$, and $\text{uAss}^{\mathcal{V}}$, are excellent.*

This follows from Propositions 2.2, 2.3, and 2.4, together with the push-out product axiom.

Proposition 3.10. *If \mathcal{O} is an excellent operad in \mathcal{V} and $\psi: A \rightarrow B$ is a cofibration of \mathcal{O} -algebras in \mathcal{C} such that A has underlying cofibrant object, then ψ is a cofibration in \mathcal{C} . In particular B also has an underlying cofibrant object*

Proof. As in the proof of Proposition 3.5, we can assume that $\psi = f'$ in (1-1) with f a cofibration between cofibrant objects in \mathcal{C} . By [8, §8] (corrected version), $A \rightarrow B$ is a transfinite composition of push-outs of the maps $\mathcal{O}_A(t)(f, \dots, f)$. These maps are cofibrations by Definition 3.8 (1). \square

Theorem 3.11. *If \mathcal{O} is an excellent operad in \mathcal{V} and*

$$\begin{array}{ccc}
 A & \xrightarrow{\psi} & B \\
 \varphi \downarrow \sim & \text{push} & \downarrow \varphi' \\
 C & \xrightarrow{\psi'} & C \cup_A B
 \end{array}$$

is a push-out of \mathcal{O} -algebras in \mathcal{C} such that the underlying objects of A and C are cofibrant, then φ' is a weak equivalence.

Proof. As in the proof of Proposition 3.5, it is enough to consider the case where $\psi = f'$ is a push-out (1-1) with f a cofibration between cofibrant objects. In this case, the map ψ' fits into the following push-out

$$\begin{array}{ccc} \mathcal{F}_O(Y) & \xrightarrow{\mathcal{F}_O(f)} & \mathcal{F}_O(Z) \\ \text{adjoint of } \varphi\bar{g} \downarrow & \text{push} & \downarrow \text{adjoint of } \varphi'\bar{g}' \\ C & \xrightarrow{\psi'} & C \cup_A B \end{array}$$

and φ' is the colimit of a sequence where all objects are cofibrant and horizontal maps are cofibrations, see Proposition 3.10 and its proof,

$$\begin{array}{ccccccc} A = B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_{t-1} \longrightarrow B_t \longrightarrow \cdots \\ \varphi \downarrow \sim & & \downarrow \varphi'_0 & & \downarrow \varphi'_1 & & \downarrow \varphi'_{t-1} & & \downarrow \varphi'_t \\ C = D_0 & \longrightarrow & D_1 & \longrightarrow & \cdots & \longrightarrow & D_{t-1} \longrightarrow D_t \longrightarrow \cdots \end{array}$$

The map φ'_t is obtained by taking horizontal push-outs in the following commutative diagram of cofibrant objects

$$\begin{array}{ccccc} B_{t-1} & \xleftarrow{\psi_t^A} & \bullet & \xrightarrow{\mathcal{O}_A(t)(f, \dots, f)} & \bullet \\ \varphi'_{t-1} \downarrow & & \downarrow & & \downarrow \\ D_{t-1} & \xleftarrow{\psi_t^C} & \bullet & \xrightarrow{\mathcal{O}_C(t)(f, \dots, f)} & \bullet \end{array}$$

Here, the two arrows pointing \rightarrow are cofibrations between cofibrant objects since \mathcal{O}_A and \mathcal{O}_C are cofibrant in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ and f is a cofibration between cofibrant objects. The second square is the map $\mathcal{O}_\varphi(t)(f, \dots, f): \mathcal{O}_A(t)(f, \dots, f) \rightarrow \mathcal{O}_C(t)(f, \dots, f)$ in $\text{Mor}(\mathcal{C})$. This map is a weak equivalence because \mathcal{O}_φ is a weak equivalence between cofibrant objects. Now the result follows as in the proof of Proposition 3.6. \square

Cofibrant functors preserve weak equivalences between cofibrant objects.

Lemma 3.12. *Any functor $F: \mathcal{N}^n \rightarrow \mathcal{M}$ which is cofibrant in $\mathcal{M}^{\mathcal{N}^n}$ takes a weak equivalence between cofibrant objects in \mathcal{N}^n , with respect to the product model structure, to a weak equivalence between cofibrant objects in \mathcal{M} .*

Proof. Given trivial cofibrations between cofibrant objects $g_i: Y_i \rightarrow Z_i$ in \mathcal{N} , $1 \leq i \leq n$, the natural map $F(Y_1, \dots, Y_n) \rightarrow F(g_1, \dots, g_n)$ in $\mathcal{M}^{\mathbf{2}^n}$ from the constant diagram is a trivial cofibration since the latching map of $F(g_1, \dots, g_n)$ at any object different from $(0, \dots, 0)$ is a trivial cofibration. This trivial cofibration in $\mathcal{M}^{\mathbf{2}^n}$, evaluated at $(1, \dots, 1) \in \mathbf{2}^n$, induces a trivial cofibration $F(Y_1, \dots, Y_n) \rightarrow F(Z_1, \dots, Z_n)$ in \mathcal{M} [5, Proposition 15.3.11 (1)]. This proves that F preserves trivial cofibrations between cofibrant objects. The result now follows from Ken Brown’s lemma [6, Lemma 1.1.12]. \square

As a consequence, weak equivalences between cofibrant objects in big functor categories are closed under horizontal composition.

Corollary 3.13. *Given weak equivalences between cofibrant objects $\tau: F \rightarrow G$ in $\mathcal{M}^{\mathcal{M}^p}$ and $\tau': F' \rightarrow G'$ in $\mathcal{M}^{\mathcal{M}^q}$, the composite $\tau^{(i-1, \cdot, \cdot, \cdot)} \tau'^{(\cdot, \cdot, p, \cdot)}: F^{(i-1, \cdot, \cdot, \cdot)} \rightarrow G^{(i-1, \cdot, \cdot, \cdot)}$ is a weak equivalence in $\mathcal{M}^{\mathcal{M}^{p+q-1}}$ for any $1 \leq i \leq p$ and $q \geq 0$.*

Colimit preserving cofibrant objects in big functor categories are closed under composition.

Lemma 3.14. *Given cofibrant objects F in $\mathcal{M}^{\mathcal{M}^p}$ and G in $\mathcal{M}^{\mathcal{M}^p}$, if F preserves colimits in the i^{th} variable then $F^{(i-1, \cdot, \cdot, \cdot)}$ is cofibrant in $\mathcal{M}^{\mathcal{M}^{p+q-1}}$, $1 \leq i \leq p$, $q \geq 0$.*

Proof. This follows from the fact that, given maps $g_j: Y_j \rightarrow Z_j$ in \mathcal{M} , $1 \leq j \leq p+q-1$, the latching map of $F(g_1, \dots, g_{i-1}, G(g_i, \dots, g_{i+q-1}), g_{i+q}, \dots, g_{p+q-1})$ at $x = (x_1, \dots, x_{p+q-1}) \in \mathbf{2}^{p+q-1}$ is the latching map of $F(g_1, \dots, g_{i-1}, h, g_{i+q}, \dots, g_{p+q-1})$ at $(x_1, \dots, x_{i-1}, 1, x_{i+q}, \dots, x_{p+q-1}) \in \mathbf{2}^p$, where h denotes the latching map of $G(g_i, \dots, g_{i+q-1})$ at $(x_i, \dots, x_{i+q-1}) \in \mathbf{2}^q$. \square

Proposition 3.15. *If $f': \mathcal{O} \rightarrow \mathcal{P}$ is a cofibration in $\text{Op}(\mathcal{V})$ and \mathcal{O} is an excellent operad such that $\mathcal{O}(n)$ is cofibrant for all $n \geq 0$, then so is \mathcal{P} .*

Proof. Like in the proof of Proposition 3.5, it is enough to assume that f' fits into a push-out diagram (2-5) where f is a sequence of cofibrations between cofibrant objects. The objects $\mathcal{P}(n)$ are cofibrant by [9, Corollary 3.7]. If A is a \mathcal{P} -algebra with underlying cofibrant object then, by Proposition 2.7, \mathcal{P}_A is the colimit of

$$\mathcal{O}_A = \mathcal{P}_{A,0} \xrightarrow{\Phi_1} \mathcal{P}_{A,1} \rightarrow \dots \rightarrow \mathcal{P}_{A,t-1} \xrightarrow{\Phi_t} \mathcal{P}_{A,t} \rightarrow \dots$$

Here \mathcal{O}_A is cofibrant in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ since \mathcal{O} is excellent. The map Φ_t is a push-out of $\tilde{\Phi}_t$, which is a cofibration between cofibrant objects in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ by Lemma 3.14 and the push-out product axiom, since f is a sequence of cofibrations between cofibrant objects and \mathcal{O}_A is cofibrant. We deduce that Φ_t is a cofibration for all $t \geq 1$, $\mathcal{P}_{A,t}$ is cofibrant for $t \geq 0$, the transfinite composition of the previous diagram is also a cofibration, and \mathcal{P}_A is cofibrant.

If $\varphi: A \xrightarrow{\sim} C$ is a weak equivalence of \mathcal{P} -algebras with underlying cofibrant objects in \mathcal{C} , then \mathcal{P}_φ is the colimit of a diagram,

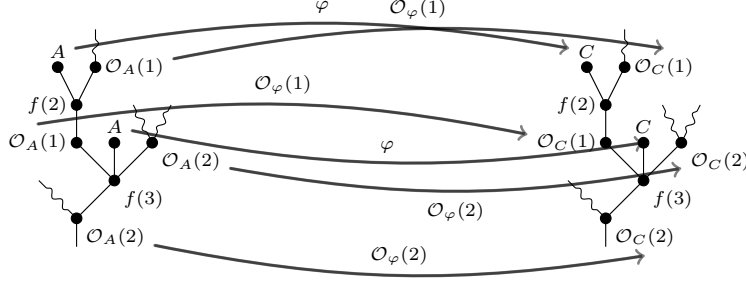
$$\begin{array}{ccccccc} \mathcal{O}_A = \mathcal{P}_{A,0} & \xrightarrow{\Phi_1^A} & \mathcal{P}_{A,1} & \rightarrow \dots & \rightarrow \mathcal{P}_{A,t-1} & \xrightarrow{\Phi_t^A} & \mathcal{P}_{A,t} \rightarrow \dots \\ \mathcal{O}_\varphi \downarrow \sim & & \downarrow \mathcal{P}_{\varphi,0} & & \downarrow \mathcal{P}_{\varphi,1} & & \downarrow \mathcal{P}_{\varphi,t-1} & & \downarrow \mathcal{P}_{\varphi,t} \\ \mathcal{O}_C = \mathcal{P}_{C,0} & \xrightarrow{\Phi_1^C} & \mathcal{P}_{C,1} & \rightarrow \dots & \rightarrow \mathcal{P}_{C,t-1} & \xrightarrow{\Phi_t^C} & \mathcal{P}_{C,t} \rightarrow \dots \end{array}$$

such that $\mathcal{P}_{\varphi,t}$ is obtained by taking horizontal push-outs in the commutative diagram

$$\begin{array}{ccccc} \mathcal{P}_{A,t-1} & \xleftarrow{\Psi_t^A} & \bullet & \xleftarrow{\tilde{\Phi}_t^A} & \bullet \\ \mathcal{P}_{\varphi,t-1} \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_{C,t-1} & \xleftarrow{\Psi_t^C} & \bullet & \xleftarrow{\tilde{\Phi}_t^C} & \bullet \end{array}$$

Aritywise, the commutative square on the right is the factor-preserving map between coproducts $\tilde{\Phi}_t^A(n) \rightarrow \tilde{\Phi}_t^C(n)$ in $\text{Mor}(\mathcal{C})^{\mathcal{C}^n}$ defined by \mathcal{O}_φ on odd inner

vertices, e.g.



Hence $\tilde{\Phi}_t^A \rightarrow \tilde{\Phi}_t^C$ is a weak equivalence by Corollary 3.13. Now this result follows as in the last paragraph of Proposition 3.6. \square

Remark 3.16. In some results above, we have assumed that tensor units are cofibrant or closely related things, e.g. that cofibrant operads have underlying cofibrant sequences. These hypotheses can be relaxed in the following way. Proofs are essentially the same, mutatis mutandis, using the results on pseudo-cofibrant and \mathbb{I} -cofibrant objects in [9, Appendices A and B], which show that these kinds of objects share many properties with cofibrant objects, assuming certain axioms when necessary. Cofibrant and \mathbb{I} -cofibrant objects are always pseudo-cofibrant. Cofibrant and pseudo-cofibrant objects coincide if and only if the tensor unit is cofibrant [9, Lemma A.7]. We recently learnt that pseudo-cofibrant objects were previously introduced in [7], where they are called semicofibrant.

If \mathcal{M} is a monoidal model category, an object F in $\mathcal{M}_S^{2^n}$ is said to be \mathbb{I} -cofibrant or pseudo-cofibrant if there is a cofibration $X \rightarrow F$ from the constant diagram on an \mathbb{I} -cofibrant or pseudo-cofibrant object X in \mathcal{M} . This is equivalent to say that $F(0, \dots, 0)$ is \mathbb{I} -cofibrant or pseudo-cofibrant and that the relative latching map of F at any $(x_1, \dots, x_n) \in 2^n$, $(x_1, \dots, x_n) \neq (0, \dots, 0)$, is a cofibration, and moreover a trivial cofibration if $x_i = 1$ for some $i \in S$. The values and latching objects of F are \mathbb{I} -cofibrant or pseudo-cofibrant in \mathcal{M} . Under the strong unit axiom, any weak equivalence in $\mathcal{M}_S^{2^n}$ between pseudo-cofibrant objects induces weak equivalences between latching objects. We define \mathbb{I} -cofibrant and pseudo-cofibrant objects in big functor categories $\mathcal{M}^{\mathcal{N}^n}$, $n \geq 0$, and $\mathcal{M}^{\mathcal{N}^{(\mathbb{N})}}$ as in Definition 3.3.

We can define *strong* homotopical notions in big functor categories, with \mathcal{N} a monoidal model category, by just requiring the sources (and hence the targets) of the cofibrations g_i in Definition 3.3 to be cofibrant or \mathbb{I} -cofibrant. Moreover, we can define *very strong* homotopical notions by more generally allowing the sources (and hence the targets) of the cofibrations g_i to be pseudo-cofibrant. For instance, a map is a strong (resp. very strong) weak equivalence in $\mathcal{M}^{\mathcal{N}^n}$ if and only if it yields a weak equivalence in \mathcal{M} when evaluated at n (\mathbb{I} -)cofibrant (resp. pseudo-cofibrant) objects in \mathcal{N} .

If \mathcal{M} is a monoidal model category, the identity functor is strongly \mathbb{I} -cofibrant and very strongly pseudo-cofibrant in $\mathcal{M}^{\mathcal{M}}$. These properties are shared by the n -fold tensor product in $\mathcal{M}^{\mathcal{M}^n}$, $n \geq 2$, by the push-out product axiom. Moreover, if Y is cofibrant/ \mathbb{I} -cofibrant/pseudo-cofibrant in \mathcal{M} then the functor $(X_1, \dots, X_n) \mapsto Y \otimes \bigotimes_{i=1}^n X_i$ is very strongly cofibrant/strongly \mathbb{I} -cofibrant/very strongly pseudo-cofibrant in $\mathcal{M}^{\mathcal{M}^n}$, $n \geq 0$. Furthermore, if f is a (trivial) cofibration in \mathcal{M} then

the natural transformation $(X_1, \dots, X_n) \mapsto f \otimes \bigotimes_{i=1}^n X_i$ is a very strong (trivial) cofibration, $n \geq 0$. If f is a weak equivalence between pseudo-cofibrant objects in \mathcal{M} and \mathcal{M} satisfies the strong unit axiom, then the previous natural transformation is a very strong weak equivalence.

In Proposition 3.5, if we assume that the objects $z(\mathcal{O}(n))$ are pseudo-cofibrant in \mathcal{C} , $n \geq 0$, we derive that the enveloping functor-operad \mathcal{O}_A of a cofibrant \mathcal{O} -algebra A is very strongly pseudo-cofibrant in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$, and that any cofibration $A \rightarrow B$ between cofibrant \mathcal{O} -algebras induces a very strong cofibration between the enveloping functor-operads $\mathcal{O}_A \rightarrow \mathcal{O}_B$. Moreover, if $A \rightarrow B$ fits into a push-out (1-1) with f a cofibration between pseudo-cofibrant objects then the objects $\mathcal{O}_{B,t}$ in Proposition 2.8 are also pseudo-cofibrant. A sufficient condition is that \mathcal{O} is a sequence of (\mathbb{I}) -cofibrant objects in \mathcal{V} , since z is a strong monoidal left Quillen functor. If, moreover, $\mathcal{O}(n)$ is cofibrant (resp. \mathbb{I} -cofibrant) for a certain $n \geq 0$ then $\mathcal{O}_A(n)$ is very strongly cofibrant (resp. strongly \mathbb{I} -cofibrant) in $\mathcal{C}^{\mathcal{C}^n}$. This applies to the associative operad, which is cofibrant in arity 0 and \mathbb{I} -cofibrant in higher arities. It also applies to any cofibrant operad \mathcal{O} , since $\mathcal{O}(1)$ is \mathbb{I} -cofibrant and $\mathcal{O}(n)$ is cofibrant for all $n \neq 1$, see [9, Corollary C.3].

In Proposition 3.6, ϕ_{η_A} is actually a very strong weak equivalence. Moreover, if \mathcal{C} satisfies the strong unit axiom then it is enough to assume that $z(\phi): z(\mathcal{O}) \rightarrow z(\mathcal{P})$ is a sequence of weak equivalences between pseudo-cofibrant objects. This condition on $z(\phi)$ holds if $\phi: \mathcal{O} \rightarrow \mathcal{P}$ is a weak equivalence between operads with underlying (\mathbb{I}) -cofibrant sequences and z satisfies the \mathbb{I} -cofibrant axiom (which says that z preserves weak equivalences between (\mathbb{I}) -cofibrant objects).

The transfer of (very) strong homotopical notions in big functor categories along Quillen pairs $L: \mathcal{M} \rightleftarrows \mathcal{N}: R$ between monoidal model categories works as follows. The ‘functor’ $\mathcal{M}^{\mathcal{M}^n} \rightarrow \mathcal{N}^{\mathcal{M}^n}$ induced by left composition with L preserves (very) strong cofibrations and trivial cofibrations. This only uses that L is a left Quillen functor. If L satisfies the pseudo-cofibrant axiom, then L takes \mathbb{I} -cofibrant objects to pseudo-cofibrant objects, so left composition with L takes (very) strong \mathbb{I} -cofibrant objects to (very) strong pseudo-cofibrant objects. Moreover, under the pseudo-cofibrant axiom, right composition with cartesian powers of L , $\mathcal{N}^{\mathcal{M}^n} \rightarrow \mathcal{N}^{\mathcal{M}^n}$, takes each very strong homotopical notion to the corresponding strong homotopical notion. This is enough to ensure that Proposition 3.7 remains true if we replace the cofibrancy condition on the tensor units of \mathcal{V} and \mathcal{W} with the requirement that \mathcal{V} , \mathcal{W} , \mathcal{C} and \mathcal{D} satisfy the strong unit axiom and F , \bar{F} , $z_{\mathcal{C}}$ and $z_{\mathcal{D}}$ satisfy the pseudo-cofibrant and \mathbb{I} -cofibrant axioms. We actually obtain that $\chi_{\mathcal{O},A}$ is a strong weak equivalence.

An operad \mathcal{O} is *very strongly pseudo-excellent* if the enveloping functor-operad \mathcal{O}_A of an \mathcal{O} -algebra A with underlying pseudo-cofibrant object is very strongly pseudo-cofibrant in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$ and any weak equivalence between such \mathcal{O} -algebras $\varphi: A \rightarrow C$ induces a very strong weak equivalence $\mathcal{O}_{\varphi}: \mathcal{O}_A \rightarrow \mathcal{O}_C$ in $\mathcal{C}^{\mathcal{C}^{(\mathbb{N})}}$. This notion is neither stronger nor weaker than the notion of excellent operad. Under the strong unit axiom, the operads in Proposition 3.9 are also very strongly pseudo-excellent. Proposition 3.10 holds if \mathcal{O} is very strongly pseudo-excellent and the \mathcal{O} -algebra A has an underlying pseudo-cofibrant object (B would just be pseudo-cofibrant in this case). Theorem 3.11 is true if \mathcal{C} satisfies the strong unit axiom, \mathcal{O} is very strongly pseudo-excellent, and the underlying objects of A and C are just pseudo-cofibrant.

The pseudo-cofibrant version of Lemma 3.12 is more involved. If \mathcal{C} satisfies the strong unit axiom, $F: \mathcal{N}^n \rightarrow \mathcal{M}$ is very strongly pseudo-cofibrant in $\mathcal{M}^{\mathcal{N}^n}$ and, for $\tilde{\mathbb{I}}$ a cofibrant replacement of the tensor unit, we have a natural isomorphism $F(\tilde{\mathbb{I}} \otimes X_1, \dots, \tilde{\mathbb{I}} \otimes X_n) \cong \tilde{\mathbb{I}}^{\otimes n} \otimes F(X_1, \dots, X_n)$, then F takes weak equivalences between pseudo-cofibrant objects $g_i: Y_i \rightarrow Z_i$, $1 \leq i \leq n$, to a weak equivalence between pseudo-cofibrant objects $F(Y_1, \dots, Y_n) \rightarrow F(Z_1, \dots, Z_n)$. These hypotheses are satisfied by the components of the enveloping functor-operad of any \mathcal{O} -algebra A with underlying pseudo-cofibrant object in \mathcal{C} , provided the operad \mathcal{O} is very strongly pseudo-excellent. Let us check the previous statement. The functor $\tilde{\mathbb{I}} \otimes F$ is very strongly cofibrant in $\mathcal{M}^{\mathcal{N}^n}$, in particular it is cofibrant, so $\tilde{\mathbb{I}} \otimes F(\tilde{\mathbb{I}} \otimes g_1, \dots, \tilde{\mathbb{I}} \otimes g_n) \cong \tilde{\mathbb{I}}^{\otimes(n+1)} \otimes F(g_1, \dots, g_n)$ is a weak equivalence. Since $\tilde{\mathbb{I}}^{\otimes(n+1)}$ is also a cofibrant replacement of the tensor unit, we deduce that $F(g_1, \dots, g_n)$ is also a weak equivalence. Its source and target are pseudo-cofibrant because F is very strongly pseudo-cofibrant, so it takes pseudo-cofibrant values when evaluated at pseudo-cofibrant objects. The pseudo-version of Corollary 3.13 also holds for very strong homotopical notions under these extra assumptions.

The very strongly pseudo-cofibrant version of Lemma 3.14 is obviously valid. Finally, Proposition 3.15 also holds for very strongly pseudo-excellent operads provided \mathcal{C} satisfies the strong unit axiom. It actually suffices that the objects $\mathcal{O}(n)$ are (\mathbb{I}) -cofibrant, $n \geq 0$.

REFERENCES

- [1] M. Batanin and M. Weber, *Algebras of higher operads as enriched categories*, Appl. Categ. Structures **19** (2011), no. 1, 93–135.
- [2] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831.
- [3] B. Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics, vol. 1967, Springer-Verlag, Berlin, 2009.
- [4] J. E. Harper, *Homotopy theory of modules over operads and non- Σ operads in monoidal model categories*, J. Pure Appl. Algebra **214** (2010), no. 8, 1407–1434.
- [5] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.
- [6] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.
- [7] L. G. Lewis Jr. and M. A. Mandell, *Modules in monoidal model categories*, J. Pure Appl. Algebra **210** (2007), no. 2, 395–421.
- [8] F. Muro, *Homotopy theory of nonsymmetric operads*, Algebr. Geom. Topol. **11** (2011), 1541–1599.
- [9] ———, *Homotopy theory of non-symmetric operads, II: Change of base category and left properness*, Algebr. Geom. Topol. **14** (2014), 229–281.
- [10] S. Schwede and B. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511.

UNIVERSIDAD DE SEVILLA, FACULTAD DE MATEMÁTICAS, DEPARTAMENTO DE ÁLGEBRA, AVDA. REINA MERCEDES S/N, 41012 SEVILLA, SPAIN

E-mail address: `fmuro@us.es`

URL: `http://personal.us.es/fmuro`